PROPAGATION OF REGULARITY AND DECAY OF SOLUTIONS TO NONLINEAR DISPERSIVE EQUATIONS (DRAFT 0)

FELIPE LINARES AND GUSTAVO PONCE

1. Introduction

In these notes we shall discuss some special properties of solutions to nonlinear dispersive equations. These properties are concerned with the propagation of regularity and decay of their solutions on the line $\mathbb{R}$ or in higher dimension.

To describe it let us consider first the initial value problem (IVP) associated to a nonlinear dispersive equation,

$$\begin{cases} \partial_t v + \phi(-i\partial_x)v = F(v, \partial_x v), \quad x \in \mathbb{R},\ t > 0, \\ v(x,0) = v_0(x) \end{cases} \tag{1.1}$$

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where ϕ is real valued mensurable function and F is a nonlinear function of v and it first order space derivative.

question **Question 1.** *Assuming that the IVP* (1.1) *is locally well-posed for initial data in Sobolev spaces $H^s(\mathbb{R})$, $s > s_0$, for some $s_0 \in \mathbb{R}$. If the initial data v_0 possesses an extra regularity, say, $v_0 \in H^l((\alpha,\infty))$, for some $\alpha \in \mathbb{R}$ and $l > s_0$, what can be said about the solution $v(x,t)$?*

We shall see that in this case the regularity moves with infinite speed to its left as time evolves. This phenomena was first observed in [47] in solutions of the IVP associated to the generalized Korteweg-de Vries (KdV) equation,

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k\,\partial_x u = 0, \quad x,t \in \mathbb{R},\ \ k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

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Our main goal is to describe the propagation of regularity property present in solutions of several nonlinear dispersive equations in one or more dimensions and some consequences of this property. We shall sketch the proofs of some key results and comment on some open problems. Examples sharing

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the property briefly illustrated above include the generalized Korteweg-de Vries (KdV) equation, generalized Benjamin-Ono (BO) equation, generalized dispersion Benjamin-Ono equation, Intermediate Long Wave (ILW) equation, fifth order KdV equation, quasilinear KdV type equations and in higher dimensions the Kadomstsev-Petviashvilli (KP) equation and the Zakharov-Kusnetsov (ZK) equation.

To illustrate this general principle we shall start by studying the IVP associated to KdV equation, i.e.$k = 1$ in (1.2).

Since we are going to deal with solutions of the IVP we first introduce the notion of local well-posedness In this regard, we follow the definition introduced by Kato [55] which says that the IVP (1.2) is locally well-posed (LWP) in the function space X if given any datum $u_0 \in X$ there exist $T > 0$ and a unique solution

$$u \in C([-T,T] : X) \cap \dots.$$

of the IVP (1.2) with the map data-solution, $u_0 \to u$, being continuous. In the cases where T can be taken arbitrarily large, one says that the problem is globally well-posed (GWP). In both cases, the solution flow defines a dynamical system on X.

For the IVP (1.2) the natural function space is the classical Sobolev family

$$H^s(\mathbb{R}) = (1-\partial_x^2)^{-s/2} L^2(\mathbb{R}), \qquad s \in \mathbb{R}. \tag{1.3}$$ $\boxed{\texttt{sob}}$

In (1.2) the cases $k = 1$ and $k = 2$ in (1.2) correspond to the KdV and modified KdV (mKdV) equations, respectively. They arise as a model in nonlinear wave propagation in a shallow channel and later as models in several other physical phenomena (see [89] and references therein). Also they have been shown to be completely integrable. In particular, they possess infinitely many conservation laws. In the case of powers $k = 3, 4, \dots$ in (1.2) the corresponding solutions satisfy just three conservation laws.

The problem of finding the minimal regularity on the data u_0 in the Sobolev scale in (1.3) which guarantees that the IVP (1.2) is well posed has been studied extensively. The list of reference is long and we will cite the most relevant results in this direction, [100], [7], [55], [61], [10], [62], [16], [18], [108], [38], [39], [64] and [63]. We shall mention that the best result using contraction mapping arguments is global well-posedness in $H^s(\mathbb{R})$, $s \geq -\frac{3}{4}$. Recently, in [63] it was established that the IVP for the KdV is globally well-posed in $H^{-1}(\mathbb{R})$ by using an argument based on the integrability condition of the KdV equation.

We observe that in [83] it was shown that a local solution of the IVP (1.2) with $k = 4$ corresponding to smooth initial data can develop singularities in finite time.

One of the key properties in the study of local well-posedness for the IVP associated to the KdV has been the so-called Kato's smoothing effect. More precisely, Kato in [55] showed that solutions of the IVP (1.2), $k = 1$, satisfies that

$$\int_0^T \int_{-R}^{R} |\partial_x u(x,t)|^2 \, dxdt \leq c(R;T;\|u_0\|_{L^2}). \tag{1.4}$$ kato's

Roughly, this tells us that an initial data $u_0 \in L^2(\mathbb{R})$ the corresponding solution satisfies that $u \in H^1_{\text{loc}}(\mathbb{R})$ for almost every time t. Extensions of this properties has been obtained in [71] and [60]. Moreover, it has been shown to be an intrinsic in nonlinear dispersive models.

We shall point out that the proof of (1.4) given by Kato relies on weighted energy estimates with a weight only depending on the x-variable. This technique will we used in the arguments of the results in these notes.

The well-posedness of the IVP (1.2) has been also studied in the weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^r dx), \quad s, r \in \mathbb{R}. \tag{1.5}$$ wei-sob

In [55] it was established the LWP of the IVP (1.2) in $Z_{s,r}$ with r an even integer and $s \geq r$. In particular, this implies the well-posedness of the IVP (1.2) in the Schwartz space $\mathscr{S}(\mathbb{R})$. The proof of this result is based in the commutative relation of the operators

$$\mathscr{L} = \partial_t + \partial_x^3 \quad \text{and} \quad \Gamma = x - 3t\partial_x^2.$$

The above result was extended to the cases $r > 0$, $s \geq r$, r a real number, in [90] and [27]. In particular, in [46] it was shown that the hypothesis $s \geq r$ is necessary. More precisely, it was established in [46] that if $u \in C([-T,T] : H^s(\mathbb{R})) \cap \dots$ is a solution of the IVP (1.2) with $s \geq \max\{s_k; 0\}$ (with s_k as above) and there exist two different times $t_1, t_2 \in [-T,T]$ such that $|x|^\alpha u(\cdot,t_j) \in L^2(\mathbb{R})$ for $j = 1,2$ with $2\alpha > s$, then $u \in C([-T,T] : H^{2\alpha}(\mathbb{R}))$.

Thus, if $u_0 \in Z_{s,r}$ with $r > s$, then at time $t \neq 0$ the solution $u(\cdot,t)$ stays only in $Z_{s,s}$, i.e. the extra decay $r - s$ is not preserved by the solution flow. As a consequence of our results obtained here we shall see that this extra-decay, no preserved by the flow, is transformed into extra regularity in a precise manner (see Corollary 2.11).

The results above are concerned with regularity and decay properties of solutions of the IVP (1.2) in symmetric spaces. We are going now to briefly discuss regularity results for asymmetric weighted spaces. In this case, the resultant properties have to be restricted to forward times $t>0$, one has the following result found in [55] for the KdV equation, $k=1$ in (1.2), in the space $L^2(e^{\beta x}dx)$, $\beta>0$. In [55] it was shown that the persistence property holds for L^2-solutions in $L^2(e^{\beta x}dx)$, $\beta>0$, for $t>0$. Moreover, formally in this space the operator $\partial_t+\partial_x^3$ becomes $\partial_t+(\partial_x-\beta)^3$ so the solutions of the equation exhibit a parabolic behavior. More precisely, the following result for the KdV equation was proven in [55] (Theorem 11.1 and Theorem 12.1).

thm1 **Theorem 1.1** ([55]). *Let $u\in C([0,\infty):H^2(\mathbb{R}))$ be a solution of the IVP* (1.2) *with $k=1$. If*

$$u_0\in H^2(\mathbb{R})\cap L^2(e^{\beta x}dx),\quad \text{for some }\ \beta>0, \tag{1.6}$$ 1.3

then

$$e^{2\beta x}u\in C([0,\infty):L^2(\mathbb{R}))\cap C((0,\infty):H^{\infty}(\mathbb{R})). \tag{1.7}$$ 1.4

Moreover, the map data-solution $u_0\to u$ is continuous from $L^2(\mathbb{R})\cap L^2(e^{\beta x}dx)$ to $C([0,T]:L^2(e^{\beta x}dx))$, for any $T>0$.

It is easy to see that the result of Theorem 1.1 extends to solutions of the IVP (1.2) for any $k\in\mathbb{Z}^+$ in their positive time interval of existence $[0,T]$.

In the same regard one has the following result established in [108] for the KdV equation, i.e. $k=1$ in (1.2):

13 **Theorem 1.2** ([108]). *If u_1 is the solution of the IVP associated to the KdV equation, i.e.* (1.2) *$k=1$, with data $u_1(x,0)=u_0\in L^2(e^{\delta|x|^{1/2}}dx)$, $\delta>0$, then $u_1(x,t)$ becomes analytic in x for each $t\neq 0$.*

A3 **Remark 1.3.**

(1) *The proof of Theorem 1.2 is based on the inverse scattering method. In particular, a similar result is unknown for generalized KdV equation* (1.2) *with $k=2,3,\dots$.*

(2) *In* [97] *Theorem 1.2 was obtained for $t>0$ under the decay assumption restricted to $x>0$.*

In view of the results in Theorem 1.1 is natural to ask what is the strongest weighted space where persistence of the solutions of (1.2) holds. The following uniqueness result obtained in [23] gives an upper bound of this weight.

thm2 **Theorem 1.4** ([23]). *There exists $c_0=c_0(k)>0$ such that for any pair*

$$u_1,\,u_2\in C([0,1]:H^4(\mathbb{R})\cap L^2(|x|^2dx))$$

of solutions of (1.2)*, if*

$$u_1(\cdot,0)-u_2(\cdot,0),\;\; u_1(\cdot,1)-u_2(\cdot,1)\in L^2(e^{c_0x_+^{3/2}}\,dx), \tag{1.8}$$ 3:2

then $u_1\equiv u_2$.

Thus, by taking $u_2\equiv 0$ one gets a restriction on the possible decay of a non-trivial solution of (1.2) at two different times. It was established in [46] that this result is optimal. More precisely, the following was proven in [46]:

thm3 **Theorem 1.5** ([46])**.** *Let a_0 be a positive constant. For any given data*

$$u_0\in H^s(\mathbb{R})\cap L^2(e^{a_0x_+^{3/2}}\,dx),\quad s>s_k\;\; \textit{defined above} \tag{1.9}$$ 0.1

the unique solution of the IVP (1.2) *satisfies that for any* $T>0$

$$\sup_{t\in[0,T]}\int_{-\infty}^{\infty} e^{a(t)x_+^{3/2}}|u(x,t)|^2dx\le c^* \tag{1.10}$$ 0.2

with $=c^*(a_0;\|u_0\|_2;\|e^{a_0x_+^{3/2}/2}u_0\|_2;T;k)$ *and*

$$a(t)=\frac{a_0}{(1+27a_0^2t/4)^{1/2}}. \tag{1.11}$$ 0.3

In the next sections we shall describe the results concerning propagation of regularity as mentioned in (1) for solutions of the IVP associated to the KdV (1.2) ($k=1$) as well as other canonical nonlinear dispersive equations.

These notes are organized in Section 2 we shall present the results regarding propagation of regularity and decay of solutions of the IVP (1.2). We will sketch some proofs and comment its consequences. Next, we shall present extensions of these results obtained for solutions of the gKdV equation to fractional Sobolev spaces and to general quasilinear KdV type equations. In particular the latter result. In Section 3, we will deal with some nonlocal nonlinear dispersive models, these includes the Benjamin-Ono equation, the Intermediate Long Wave Equation and the dispersion generalized Benjamin-Ono equation. The propagation of regularity in higher dimensional models will be examined in Section 5. Finally, in Section 6, we will discuss some additional results and a list of some open problems.

2. KORTEWEG-DE VRIES EQUATION

In this section we will present the regularity properties for solutions of the IVP associated to the KdV in the direction of Question 1. We plan to give an sketch of the technique employed to establish such property as well as describe a series of interesting consequences.

We start by defining the class of solutions to the IVP (1.2) for which these properties apply. We shall rely on the following well-posedness result which is a consequence of the arguments deduced in [61]:

thm4 **Theorem 2.1** ([61]). *If* $u_0 \in H^{3/4^+}(\mathbb{R})$*, then there exist* $T = T(\|u_0\|_{3/4^+,2}; k) > 0$ *and a unique solution of the IVP* (1.2) *such that*

$$\begin{aligned}
&\text{(i)} && u \in C([-T,T] : H^{3/4^+}(\mathbb{R})),\\
&\text{(ii)} && \partial_x u \in L^4([-T,T] : L^\infty(\mathbb{R})), \quad (Strichartz),\\
&\text{(iii)} && \sup_x \int_{-T}^{T} |J^r \partial_x u(x,t)|^2\, dt < \infty \quad \text{for} \quad r \in [0, 3/4^+],\\
&\text{(iv)} && \int_{-\infty}^{\infty} \sup_{-T \le t \le T} |u(x,t)|^2\, dx < \infty,
\end{aligned} \tag{2.1}$$ notes-2

with $J = (1-\partial_x^2)^{1/2}$*. Moreover, the map data-solution,* $u_0 \to u(x,t)$ *is locally continuos (smooth) from* $H^{3/4+}(\mathbb{R})$ *into the class defined in* (2.1).

Our first result is concerned with the propagation of regularity in the right hand side of the data for positive times.

thm5 **Theorem 2.2** ([47]). *If* $u_0 \in H^{3/4^+}(\mathbb{R})$ *and for some* $l \in \mathbb{Z}^+$, $l \ge 1$ *and* $x_0 \in \mathbb{R}$

$$\|\,\partial_x^l u_0\|^2_{L^2((x_0,\infty))} = \int_{x_0}^{\infty} |\partial_x^l u_0(x)|^2 dx < \infty, \tag{2.2}$$ notes-3

then the solution of the IVP (1.2) *provided by Theorem 2.1 satisfies that for any* $v > 0$ *and* $\varepsilon > 0$

$$\sup_{0\le t\le T} \int_{x_0+\varepsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)\, dx < c, \tag{2.3}$$ notes-4

for $j = 0, 1, \dots, l$ *with* $c = c(l; \|u_0\|_{3/4^+,2}; \|\,\partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \varepsilon; T)$.

In particular, for all $t \in (0,T]$*, the restriction of* $u(\cdot,t)$ *to any interval* (x_0,∞) *belongs to* $H^l((x_0,\infty))$.

Moreover, for any $v \ge 0$, $\varepsilon > 0$ *and* $R > 0$

$$\int_0^T \int_{x_0+\varepsilon-vt}^{x_0+R-vt} (\partial_x^{l+1} u)^2(x,t)\, dx dt < c, \tag{2.4}$$ notes-5

with $c = c(l; \|u_0\|_{3/4^+,2}; \|\,\partial_x^l u_0\|_{L^2((x_0,\infty))}; v; \varepsilon; R; T)$.

Theorem 2.2 tells us that the H^l-regularity ($l \in \mathbb{Z}^+$) on the right hand side of the data travels forward in time with infinite speed. Notice that since the equation is reversible in time a gain of regularity in $H^s(\mathbb{R})$ cannot occur so at $t > 0$, so $u(\cdot,t)$ fails to be in $H^l(\mathbb{R})$ due to its decay at $-\infty$. In this regard, it was also shown in [47] :

cor-pr-kdv **Corollary 2.3.** *for any $\delta>0$ and $t\in(0,T)$ and $j=1,\dots,l$*

$$\int_{-\infty}^{\infty}\frac{1}{\langle x_-\rangle^{j+\delta}}(\partial_x^j u)^2(x,t)\,dx\le\frac{c}{t},$$

with $c=c(\|u_0\|_{3/4^+,2};\|\partial_x^j u_0\|_{L^2((x_0,\infty))};x_0;\delta)$, $x_-=\max\{0;-x\}$ and $\langle x\rangle=(1+x^2)^{1/2}$.

Below we will sketch the main steps of the proof of Theorem 2.2 and the proof of Corollary (2.3).

The next result describes the persistence properties and regularity effects, for positive times, in solutions associated with data having polynomial decay in the positive real line.

thm5-b **Theorem 2.4** ([47])**.** *If $u_0\in H^{3/4^+}(\mathbb{R})$ and for some $n\in\mathbb{Z}^+$, $n\ge1$,*

$$\|x^{n/2}u_0\|^2_{L^2((0,\infty))}=\int_0^{\infty}|x^n|\,|u_0(x)|^2dx<\infty, \tag{2.5}$$ b1

then the solution u of the IVP (1.2) *provided by Theorem 2.1 satisfies that*

$$\sup_{0\le t\le T}\int_0^{\infty}|x^n|\,|u(x,t)|^2\,dx\le c \tag{2.6}$$ b2

with $c=c(n;\|u_0\|_{3/4^+,2};\|x^{n/2}u_0\|_{L^2((0,\infty))};T)$.

Moreover, for any $\varepsilon,\delta,R>0, v\ge0$, $m,\ j\in\mathbb{Z}^+$, $m+j\le n$, $m\ge1$,

$$\begin{aligned}\sup_{\delta\le t\le T}&\int_{\varepsilon-vt}^{\infty}(\partial_x^m u)^2(x,t)\,x_+^j\,dx\\&+\int_\delta^T\int_{\varepsilon-vt}^{R-vt}(\partial_x^{m+1}u)^2(x,t)\,x_+^{j-1}\,dxdt\le c,\end{aligned} \tag{2.7}$$ b3

with $c=c(n;\|u_0\|_{3/4^+,2};\|x^{n/2}u_0\|_{L^2((0,\infty))};T;\delta;\varepsilon;R;v)$.

The proofs of Theorem 2.2 and Theorem 2.4 that they still hold for solutions of the "defocussing" k-gKdV

$$\partial_t u+\partial_x^3u-u^k\,\partial_x u=0,\quad x,t\in\mathbb{R},\ k\in\mathbb{Z}^+.$$

Therefore, our results apply to $u(-x,-t)$ if $u(x,t)$ is a solution of (1.2). In other words, Theorem 2.2 and Theorem 2.4 resp. remain valid, backward in time, for datum satisfying the hypothesis (2.2) and (2.5) resp. on the left hand side of the real line.

As a direct consequence of Theorem 2.2 and Theorem 2.4, the above comments and the time reversible character of the equation in (1.2) one has:

cor1 **Corollary 2.5.** *Let $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$ be a solution of the equation in* (1.2) *described in Theorem 2.1. If there exist $m \in \mathbb{Z}^+$, $\hat{t} \in (-T,T)$, $a \in \mathbb{R}$ such that*

$$\partial_x^m u(\cdot,\hat{t}) \notin L^2((a,\infty)),$$

then for any $t \in [-T,\hat{t})$ and any $\beta \in \mathbb{R}$

$$\partial_x^m u(\cdot,t) \notin L^2((\beta,\infty)), \quad \textit{and} \quad x^{m/2}\, u(\cdot,t) \notin L^2((0,\infty)).$$

Next, as a consequence of Theorem 2.2 and Theorem 2.4 one has that for an appropriate class of data the singularity of the solution travels with infinite speed to the left as time evolves. In the integrable cases $k = 1,2$ this is expected as part of the so called *resolution conjecture*, (see [22]). Also as a consequence of the time reversibility property one has that the solution cannot have had some regularity in the past.

cor2 **Corollary 2.6.** *Let $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$ be a solution of the equation in* (1.2) *described in Theorem 2.1. If there exist $n, m \in \mathbb{Z}^+$ with $m \leq n$ such that for some $a, b \in \mathbb{R}$ with $a < b$*

$$\int_b^{\infty} |\partial_x^n u_0(x)|^2 dx < \infty \quad \textit{but} \quad \partial_x^m u_0 \notin L^2((a,\infty)), \tag{2.8}$$ A

then for any $t \in (0,T)$ and any $v > 0$ and $\varepsilon > 0$

$$\int_{b+\varepsilon-vt}^{\infty} |\partial_x^n u(x,t)|^2 dx < \infty,$$

and for any $t \in (-T,0)$ and any $\alpha \in \mathbb{R}$

$$\int_{\alpha}^{\infty} |\partial_x^m u(x,t)|^2 dx = \infty.$$

Remark 2.7. *(1) If in Corollary 2.6 in addition to* (2.8) *one assumes that*

$$\int_{-\infty}^{a} |\partial_x^n u_0(x)|^2 dx < \infty, \tag{2.9}$$ Ab

then by combining the results in this corollary with the group properties it follows that

$$\int_{-\infty}^{\beta} |\partial_x^n u(x,t)|^2\, dx = \infty, \quad \textit{for any } \beta \in \mathbb{R} \textit{ and } t > 0.$$

This tell us that in general the regularity in the left hand side of the real line does not propagate forward in time.

(2) It follows from Theorem 2.2 that under the hypotheses (2.8)-(2.9) *the solution $u(\cdot,t)$ satisfies*

$$u(\cdot,t) \in H^n_{\text{loc}}(\mathbb{R}) \quad \textit{for all } t \in [-T,T]\backslash\{0\},$$

hence the local (m) singularity $\partial_x^m u_0 \notin L^2((a,b))$ *does not reappear in the future or in the past.*

(3) The same argument shows that for initial datum

$$u_0 \in L^2(|x|^n dx) \cap H^{3/4+}(\mathbb{R}) \setminus H^m(\mathbb{R})$$

with $m,n \in Z^+$, $m<n$, *the corresponding solution* $u(\cdot,t)$ *of* (1.2) *satisfies*

$$x_+^{n/2} u(\cdot,t) \in L^2((0,\infty)) \quad \text{and} \quad x_-^{m/2} u(\cdot,t) \notin L^2((-\infty,0)), \quad t \in (0,T].$$

Thus, one has that in general the decay in the left hand side of the real line does not propagate forward in time.

Analogously, in [46] one finds :

cor3 **Corollary 2.8.** *Let* $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$ *be a solution of the equation in* (1.2) *described in Theorem 2.1. If for* $j,m \in \mathbb{Z}^+$, $j<m$,

$$x_+^{m/2} u_0 \in L^2((0,\infty)), \ \text{and} \ \partial_x^j u_0(x) \notin L^2((\beta,\infty)), \ \text{for some} \ \beta \in \mathbb{R},$$

then for any $t \in (0,T]$

$$x_+^{m/2} u(\cdot,t) \in L^2((0,\infty)), \ \text{and} \ \partial_x^m u(\cdot,t) \in L^2((\alpha,\infty)), \ \ \forall \alpha \in \mathbb{R},$$

and for any $t \in [-T,0)$

$$x_+^{j/2} u(\cdot,t) \notin L^2((0,\infty)), \ \text{and} \ \partial_x^j u(\cdot,t) \notin L^2((\alpha,\infty)), \ \ \forall \alpha \in \mathbb{R}.$$

As a consequence of Theorem 2.4 we can improve the result in Theorem 1.4 in [46] in the case of a positive integer. More precisely,

cor4 **Corollary 2.9.** *Let* $u \in C([-T,T] : H^{3/4^+}(\mathbb{R}))$ *be a solution of the equation* (1.2) *described in Theorem 2.1. If there exist* $n_j \in \mathbb{Z}^+ \cup \{0\}$, $j = 1,2,3,4$, $t_0, t_1 \in [-T,T]$ *with* $t_0 < t_1$ *and* $a,b \in \mathbb{R}$ *such that*

$$\int_0^\infty |x|^{n_1} |u(x,t_0)|^2 dx < \infty \quad \text{and} \quad \int_a^\infty |\partial_x^{n_2} u(x,t_0)|^2 dx < \infty,$$

and

$$\int_{-\infty}^0 |x|^{n_3} |u(x,t_1)|^2 dx < \infty \quad \text{and} \quad \int_{-\infty}^b |\partial_x^{n_4} u(x,t_1)|^2 dx < \infty,$$

then

$$u \in C([-T,T] : H^s(\mathbb{R}) \cap L^2(|x|^r dx))$$

where

$$s = \min\{\max\{n_1;n_2\};\max\{n_3;n_4\}\} \quad \text{and} \quad r = \min\{n_1;n_3\}.$$

Remark 2.10. *(1) The improvement of Theorem 1.4 in* [46] *in the case of a positive follows by taking* $n_1 = n_3 \in \mathbb{Z}^+$ *and* $n_2 = n_4 = 0$.

(2) Although for the sake of the simplicity we shall not pursue this issue here, we remark that the results in Theorem 2.4 and Corollary 2.9 can be extended to non-integer values of the parameter n. In this case, one needs to combine the argument given below with those found in [91].

As it was mentioned above the solution flow of the IVP (1.2) does not preserve the class $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^r dx)$ when $r > s$. Our next result describes how the decay $(r-s)$, not preserved by the flow, is transformed into extra regularity of the solution.

cor7 **Corollary 2.11.** *If* $u_0 \in Z_{s,r}$ *with* $s > 3/4^+$, $r \in \mathbb{Z}^+$ *and* $r > s$. *Then the solution of the IVP* (1.2) $u \in C([-T,T] : Z_{s,s})$ *also satisfies that for any* $b > 0$

$$\int_b^\infty (\partial_x^r u(x,t))^2 dx < \infty, \qquad \textit{for any } t \in (0,T],$$

$$\int_0^\infty |x|^r |u(x,t)|^2\, dx < \infty, \qquad \textit{for any } t \in [0,T],$$

and

$$\int_{-\infty}^{-b} (\partial_x^r u(x,t))^2 dx < \infty, \qquad \textit{for any } t \in [-T,0),$$

$$\int_{-\infty}^{0} |x|^r |u(x,t)|^2\, dx < \infty, \qquad \textit{for any } t \in [-T,0].$$

Remark 2.12. *(1) Combining the results in Theorems 2.2 and 2.4 with those in* [46] *previously described one can deduce further properties of the solution* $u(x,t)$ *as a consequence of the regularity and decay assumptions on the data* u_0. *For example, the solution of the IVP* (1.2) *with data* $u_0 \in H^{3/4^+}(\mathbb{R}) \backslash H^1(\mathbb{R})$ *for which there exists* $n \in \mathbb{Z}^+$

$$\int_0^\infty |x^n|\, |u_0(x)|^2\, dx < \infty$$

one has, in addition to (2.6)–(2.7), *that for any* $M > 0$ *and any* $t > 0$

$$\int_{-\infty}^{-M} |x\, u(x,t)|^2\, dx = \infty \quad \textit{and} \quad \int_{-\infty}^{-M} |\partial_x u(x,t)|^2\, dx = \infty.$$

(2) The proof of Theorems 2.2 and 2.4 relies on weighted energy estimates and an iterative process for which the estimate (2.1) *(ii) is essential. This is a consequence of the following version of the Strichartz estimates* [102] *obtained in* [60]. *The solution of the linear IVP*

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{2.10}$$

linearIVP

is given by the group $\{U(t) : t \in \mathbb{R}\}$

$$U(t)v_0(x) = \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{\cdot}{\sqrt[3]{3t}}\right) * v_0(x).$$

where $Ai(\cdot)$ *denotes the Airy function*

$$Ai(x) = c \int_{-\infty}^{\infty} e^{ix\xi + i\xi^3/3} d\xi. \tag{2.11}$$ $\boxed{\texttt{airy}}$

The following inequality was established in [60] *: for any* $(\theta, \alpha) \in [0,1] \times [0,1/2]$

$$\|D^{\theta\alpha/2} U(t)u_0\|_{L^q(\mathbb{R}:L^p(\mathbb{R}))} \le c\|u_0\|_2, \tag{2.12}$$ $\boxed{\texttt{stric}}$

with $(q,p) = (6/\theta(1+\alpha), 2/(1-\theta))$. *In particular, by taking* $(\theta,\alpha) = (1,1/2)$ *one has*

$$(\int_{-\infty}^{\infty} \|D^{1/4}U(t)u_0\|_\infty^4 \, dt)^{1/4} \le c\|u_0\|_2, \tag{2.13}$$ $\boxed{\texttt{stric1}}$

which explains the hypothesis $s = 3/4^+$ *in Theorem 2.1 and the conclusion* (2.1) *(ii) on it.*

(3) Even for the associated linear IVP (2.10) *the results in Theorems 2.2 and 2.4 do not seem to have been appreciated before, even that in this (linear) case a different proof should follow based on estimates of the Airy function* $Ai(\cdot)$ *(see* (2.11)*) and its derivatives.*

(4) Although we will not pursue this issue here, it should be remarked that the results in Theorems 2.2 and 2.4 can be extended to include some continuity property in time. For example, in Theorem 2.2 in addition to (2.6) *one can show that for any* $t_0 \in (0,T)$ *and* $\varepsilon > 0$ *and any* $v > 0$

$$\lim_{t \to t_0} \int_{x_0+\varepsilon-vt}^{\infty} (\partial_x^j u)^2(x,t)dx = \int_{x_0+\varepsilon-vt_0}^{\infty} (\partial_x^j u)^2(x,t_0)dx.$$

In this case the proof follows by using an argument similar to that given in [7].

Sketch proof of Theorem 2.2. We shall intend to describe the method employ in our analysis. We begin by constructing a class of real functions $\chi_{0,\varepsilon,b}(x)$ for $\varepsilon > 0$ and $b \ge 5\varepsilon$ such that

$$\chi_{0,\varepsilon,b} \in C^\infty(\mathbb{R}), \ \ \chi'_{0,\varepsilon,b} \ge 0, \tag{2.14}$$ $\boxed{\texttt{a1}}$

$$\chi_{0,\varepsilon,b}(x) = \begin{cases} 0, & x \le \varepsilon, \\ 1, & x \ge b, \end{cases} \tag{2.15}$$ $\boxed{\texttt{a2}}$

therefore

$$\text{supp}\ \chi_{0,\varepsilon,b} \subseteq [\varepsilon, \infty),$$

$$\chi'_{0,\varepsilon,b}(x) \geq \frac{1}{b-3\varepsilon}\, 1_{[3\varepsilon,b-2\varepsilon]}(x), \tag{2.16}$$ a3

and

$$\operatorname{supp} \chi'_{0,\varepsilon,b}(x) \subseteq [\varepsilon, b]. \tag{2.17}$$ a4

Thus

$$\chi'_{0,\varepsilon/3,b+\varepsilon}(x) \geq c_j\, |\chi^{(j)}_{0,\varepsilon,b}(x)|, \quad \forall x \in \mathbb{R}, \ \forall j \geq 1. \tag{2.18}$$ a5

Also, if $x \in (3\varepsilon, \infty)$, then

$$\chi_{0,\varepsilon,b}(x) \geq \chi_{0,\varepsilon,b}(3\varepsilon) \geq \frac{1}{2}\, \frac{\varepsilon}{b-3\varepsilon}, \tag{2.19}$$ a6

and for any $x \in \mathbb{R}$

$$\chi'_{0,\varepsilon/3,b+\varepsilon}(x) \leq \frac{1}{b-3\varepsilon}. \tag{2.20}$$ a7

We also have that given $\varepsilon > 0$, $b \geq 5\varepsilon$ there exist c_1, $c_2 > 0$ such that

$$\begin{aligned} &\chi'_{0,\varepsilon,b}(x) \leq c_1\, \chi'_{0,\varepsilon/3,b+\varepsilon}(x)\chi_{0,\varepsilon/3,b+\varepsilon}(x), \\ &\chi'_{0,\varepsilon,b}(x) \leq c_2\, \chi_{0,\varepsilon/5,\varepsilon}(x). \end{aligned} \tag{2.21}$$ a8

We shall obtain this family $\{\chi_{0,\varepsilon,b} : \varepsilon > 0, \ b \geq 5\varepsilon\}$ by first considering $\rho \in C_0^\infty(\mathbb{R})$, $\rho(x) \geq 0$, even, with $\operatorname{supp}\rho \subseteq (-1,1)$ and $\int \rho(x)dx = 1$. Then defining

$$\nu_{\varepsilon,b}(x) = \begin{cases} 0, \quad x \leq 2\varepsilon, \\ \\ \frac{1}{b-3\varepsilon}x - \frac{2\varepsilon}{b-3\varepsilon}, \quad x \in [2\varepsilon, b-\varepsilon], \\ \\ 1, \quad x \geq b-\varepsilon, \end{cases} \tag{2.22}$$ a9

and

$$\chi_{0,\varepsilon,b}(x) = \rho_\varepsilon * \nu_{\varepsilon,b}(x), \tag{2.23}$$ a10

where $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(x/\varepsilon)$.

For any $n \in \mathbb{Z}^+$ define

$$\chi_{n,\varepsilon,b}(x) = x^n\, \chi_{0,\varepsilon,b}(x). \tag{2.24}$$ a11

Thus, for any $n \in \mathbb{Z}^+$

$$\chi'_{n+1,\varepsilon,b}(x) \geq (n+1)\chi_{n,\varepsilon,b}(x). \tag{2.25}$$ a12

Proof of Theorem 2.2. It is based on an induction argument. First, we shall prove (2.3) for $l = 1$ and $l = 2$ (to illustrate the method).

Case $l = 1$.

To simplify the exposition we restrict ourselves to the case $k = 1$ in (1.2), (KdV case), remark that from the proof blow this does not represent a loss of generality.

Formally, take partial derivative with respect to x of the equation in (1.2) and multiply by $\partial_x u \chi_{0,\varepsilon,b}(x+vt)$ to obtain after integration by parts the identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\partial_x u)^2(x,t)\chi_0(x+vt)\,dx - \underbrace{v\int(\partial_x u)^2(x,t)\chi_0'(x+vt)\,dx}_{A_1} \\
&+\frac{3}{2}\int(\partial_x^2 u)^2(x,t)\chi_0'(x+vt)\,dx - \underbrace{\frac{1}{2}\int(\partial_x u)^2(x,t)\chi_0'''(x+vt)\,dx}_{A_2} \\
&+\underbrace{\int\partial_x(u\partial_x u)\partial_x u(x,t)\chi_0(x+vt)\,dx}_{A_3} = 0
\end{aligned} \tag{2.26}$$

notes-10

where in χ_0 we omit the index ε, b (fixed).

Then after integrating in the time interval $[0,T]$ using the estimate (2.1)(iii) with $r = 0$ we have

$$\int_0^T |A_1(t)|\,dt \le v\int_0^T\int(\partial_x u)^2\chi_0'(x+vt)\,dxdt < c, \tag{2.27}$$

notes-11

since given v, ε, b, T as above there exist $c_0 > 0$ and $R > 0$ such that

$$\chi_0'(x+vt) \le c_0 1_{[-R,R]}(x), \quad \forall (x,t) \in \mathbb{R}\times[0,T].$$

The same argument shows that

$$\int_0^T |A_2(t)|\,dt \le c_0. \tag{2.28}$$

notes-12

Finally, since

$$\begin{aligned}
A_3 &= \int \partial_x u\,\partial_x u\,\partial_x u\,\chi_0\,dx + \int u\partial_x^2 u\,\partial_x u\,\chi_0\,dx \\
&= \frac{1}{2}\int\partial_x u\,\partial_x u\,\partial_x u\,\chi_0(x+vt)\,dx - \frac{1}{2}\int u\partial_x u\,\partial_x u\,\chi_0'(x+vt)\,dx \\
&= A_{31} + A_{32}
\end{aligned} \tag{2.29}$$

notes-13

one has

$$|A_{31}| \le \|\partial_x u\|_\infty \int(\partial_x u)^2\,\chi_0(x+vt)\,dx \tag{2.30}$$

notes-14

and

$$|A_{32}| \le \|u(t)\|_\infty \int(\partial_x u)^2\,\chi_0'(x+vt)\,dx. \tag{2.31}$$

notes-15

Hence by Sobolev embedding and (2.27) after integrating in the time interval $[0,T]$ one gets

$$\int_0^T |A_{32}|\,dt \leq \sup_{[0,T]} \|u(t)\|_{3/4^+} \int_0^T \int (\partial_x u)^2\, \chi_0'(x+vt)\,dxdt \leq c_0. \tag{2.32}$$

notes-16

Inserting the above information in (2.26), Gronwall's inequality and (2.1) (ii) yield the estimate

$$\sup_{[0,T]} \int (\partial_x u)^2\, \chi_{0,\varepsilon,b}(x+vt)\,dx + \int_0^T \int (\partial_x^2 u)^2\, \chi_{0,\varepsilon,b}'(x+vt)\,dxdt \leq c_0 \tag{2.33}$$

notes-17

with $c_0 = c_0(\varepsilon;b;v) > 0$ for any $\varepsilon > 0$, $b \geq 5\varepsilon$, $v > 0$, which proves the case $l = 1$.

Case $l = 2$.

We can assume that the solution $u(\cdot)$ satisfies (2.1) and (2.33) (case $l = 1$). Then formally one has the following identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (\partial_x^2 u)^2(x,t)\chi_0(x+vt)\,dx - \underbrace{v\int (\partial_x^2 u)^2(x,t)\chi_0'(x+vt)\,dx}_{A_1} \\
&+ \frac{3}{2}\int (\partial_x^3 u)^2(x,t)\chi_0'(x+vt)\,dx - \underbrace{\frac{1}{2}\int (\partial_x^2 u)^2(x,t)\chi_0'''(x+vt)\,dx}_{A_2} \\
&+ \underbrace{\int \partial_x^2(u\partial_x u)\partial_x^2 u(x,t)\,\chi_0(x+vt)\,dx}_{A_3} = 0.
\end{aligned} \tag{2.34}$$

notes-18

After integration in time we have from (2.33)

$$\int_0^T |A_1(t)|\,dt \leq |v| \int_0^T \int (\partial_x^2 u)^2\, \chi_0'(x+vt)\,dxdt \leq |v|\,c_0. \tag{2.35}$$

notes-19

Next by the construction of the $\chi_{0,\varepsilon,b}$'s one has that for $\varepsilon > 0$, $b \geq 5\varepsilon$, there exists $c > 0$ such that

$$\left|\chi_{0,\varepsilon,b}'''(x)\right| \leq c_{\varepsilon,b}\, \chi_{0,\varepsilon/3,b+\varepsilon}'(x) \qquad \forall x \in \mathbb{R}. \tag{2.36}$$

notes-20

Therefore, after integration in time using (2.33) with $(\varepsilon/2, b+\varepsilon)$ instead of (ε, b), it follows that

$$\begin{aligned}
\int_0^T |A_2(t)|\,dt &\leq \int_0^T \int (\partial_x^2 u)^2\, |\chi_{0,\varepsilon,b}'''(x+vt)|\,dxdt \\
&\leq c \int_0^T \int (\partial_x^2 u)^2\, |\chi_{0,\varepsilon/3,b+\varepsilon}'(x+vt)|\,dxdt \leq c.
\end{aligned} \tag{2.37}$$

notes-21

Finally, we have to consider A_3 in (2.34). Thus

$$\begin{aligned}
&\int \partial_x^2(u\partial_x u)\,\partial_x^2 u\,\chi_0(x+vt)\,dx\\
&= \int u\,\partial_x^3 u\,\partial_x^2 u\chi_0(x+vt)\,dx + 3\int \partial_x u\,\partial_x^2 u\,\partial_x^2 u\,\chi_0(x+vt)\,dx\\
&= \frac{5}{2}\int \partial_x u\,\partial_x^2 u\,\partial_x^2 u\,\chi_0(x+vt)\,dx - \frac{1}{2}\int u\,\partial_x^2 u\,\partial_x^2 u\,\chi_0'(x+vt)\,dx\\
&= A_{31}+A_{32}.
\end{aligned} \tag{2.38}$$

notes-22

Then

$$|A_{31}| \le c\|\partial_x u(t)\|_\infty \int (\partial_x^2 u)^2\,\chi_0(x+vt)\,dx \tag{2.39}$$

notes-23

where the last integral is the quantity to be estimated. After integration in time, (2.33) and Sobolev embedding we obtain that

$$\int_0^T |A_{32}(t)|\,dt \le \sup_{0\le t\le T}\|u(t)\|_\infty \int_0^T \int (\partial_x^2 u)^2\,\chi_0'(x+vt)\,dxdt \le c. \tag{2.40}$$

notes-24

Inserting the above information in (2.34) and using Gronwall's inequality it follows that

$$\begin{aligned}
&\sup_{0\le t\le T}\int (\partial_x^2 u)^2\,\chi_{0,\varepsilon,b}(x+vt)\,dx\\
&+\int_0^T \int (\partial_x^3 u)^2\,\chi_{0,\varepsilon,b}'(x+vt)\,dxdt \le c_0
\end{aligned} \tag{2.41}$$

notes-25

with $c_0 = c_0(\varepsilon;b;v) > 0$ for any $\varepsilon > 0$, $b \ge 5\varepsilon$, $v > 0$.

Proof of Corollary 2.3. To prove Corollary 2.3 we will apply the next lemma.

precise-1

Lemma 2.13. *Let $f \ge 0$ such that there exists $\alpha > 0$ such that for any $a > 0$*

$$\int_0^a f(t)\,dt \le c\,a^\alpha.$$

Then, for every $\varepsilon > 0$ it holds that

$$\int_0^\infty \frac{1}{\langle t\rangle^{\varepsilon+\alpha}} f(t)\,dt < \infty.$$

Proof of Lemma 2.13. By hypothesis we have

$$\int_0^1 f(t)\,dt \le c,$$

and

$$\int_{2^k}^{2^{k+1}} f(t)\,dt \leq \int_{0}^{2^{k+1}} f(t)\,dt \leq c(2^{k+1})^{\alpha} \quad \text{for any integer} \quad k \geq 0. \tag{2.42}$$

precise-2

From (2.42) it follows that

$$\frac{1}{(2^{k+1})^{\alpha+\varepsilon}} \int_{2^k}^{2^{k+1}} f(t)\,dt \leq \frac{c}{(2^{k+1})^{\varepsilon}}$$

Observing that $2^k \leq t \leq 2^{k+1}$ we deduce that

$$\frac{1}{2^{\alpha+\varepsilon}} \int_{2^k}^{2^{k+1}} \frac{1}{\langle t\rangle^{\alpha+\varepsilon}} f(t)\,dt \leq \frac{c}{(2^{k+1})^{\varepsilon}}.$$

The result follows.

□

Proof of Corollary 2.3. Following the proof of (2.2) one has that

$$\int_{-vt}^{\infty} |\partial_x u(x,t)|^2\,dx < cv$$

$$\int_{-t}^{t}\int_{0}^{1} |\partial_x^2 u(x,t)|^2\,dxdt \leq tv$$

and

$$\int_{-vt}^{\infty} |\partial_x^2 u(x,t)|^2\,dx \leq tv^2 = (tv)^2\frac{c}{t}.$$

Thus, combining these estimates with Lemma 2.13 one gets the desired result. □

3. Extensions and Quasilinear KdV type equations

Our aim in this section is to present extensions of the results in Theorem 2.2. First we we will consider the propagation of regularity for initial data with extra regularity in fractional Sobolev spaces $H^s((\alpha,\infty))$, for $s \in \mathbb{R}$ and α a real number. Second, we will be to establish propagation as in Theorem 2.2 for solutions of quasilinear KdV type equations.

We start the result concerning the propagation of regularity in fractional Sobolev spaces.

thm7 **Theorem 3.1** ([59]). *Let $u_0 \in H^{3/4^+}(\mathbb{R})$. If for some $s \in \mathbb{R}$, $s > 3/4$, and for some $x_0 \in \mathbb{R}$*

$$\| J^s u_0\|^2_{L^2((x_0,\infty))} = \int_{x_0}^{\infty} |J^s u_0(x)|^2 dx < \infty, \tag{3.1}$$

anotes-3

then the solution $u = u(x,t)$ of the IVP (1.2) *provided by Theorem 2.1 satisfies that for any $v > 0$ and $\varepsilon > 0$*

$$\sup_{0\le t\le T} \int_{x_0+\varepsilon-vt}^{\infty} (J^r u)^2(x,t)\, dx < c, \tag{3.2}$$

anotes-4

for $r \in (3/4, s]$ with $c = c(l; \|u_0\|_{3/4^+,2}; \| J^r u_0\|_{L^2((x_0,\infty))}; v; \varepsilon; T)$.

Moreover, for any $v \ge 0$, $\varepsilon > 0$ and $R > 0$

$$\int_0^T \int_{x_0+\varepsilon-vt}^{x_0+R-vt} (J^{s+1}u)^2(x,t)\, dxdt < c, \tag{3.3}$$

anotes-5

with $c = c(l; \|u_0\|_{3/4^+,2}; \| J^s u_0\|_{L^2((x_0,\infty))}; v; \varepsilon; R; T)$.

Essentially the proof of Theorem 3.1 follows the same techniques as in Theorem 2.2 discussed in the previous section. However, we need to make use of pseudo-differential operators techniques in order to deal with the Bessel operators J^s, defined via Fourier transform as

$$\widehat{J^s f}(\xi) = (1+|\xi|^2)^{s/2} \widehat{f}(\xi).$$

We present the main tools we employ to establish Theorem 3.1.

Let T_a be a pseudo-differential operator whose symbol

$$\sigma(T_a) = a(x,\xi) \in S^r, \;\; r \in \mathbb{R}, \tag{3.4}$$

symbol

so that

$$T_a f(x) = \int_{\mathbb{R}^n} a(x,\xi) \widehat{f}(\xi)\, e^{2\pi i x\cdot\xi} d\xi. \tag{3.5}$$

pseudo

The following result is the singular integral realization of a pseudo-differential operator, whose proof can be found in [101], Chapter 4.

TA3 **Theorem 3.2.** *Using the above notation* (3.4)-(3.5) *one has that*

$$T_a f(x) = \int_{\mathbb{R}^n} k(x, x-y) f(y) dy, \quad \text{if} \quad x \notin supp(f) \tag{3.6}$$

pseudo-si

where $k \in C^\infty(\mathbb{R}^n \times R^n - \{0\})$ satisfies : $\forall\, \alpha, \beta \in (Z^+)^n\ \forall N \ge 0$

$$\begin{aligned} &|\partial_x^\alpha \partial_z^\beta k(x,z)| \le A_{\alpha,\beta,N,\delta}\, |z|^{-(n+m+|\beta|+N)}, \qquad |z| \ge \delta, \\ &\text{if} \quad n+m+|\beta|+N > 0 \qquad \text{uniformly in} \;\; x \in \mathbb{R}^n. \end{aligned} \tag{3.7}$$

estimate1

We consider the one-dimensional case $x \in \mathbb{R}$ to apply in our case.

As a direct consequence of Theorem 3.2 one has :

c1 **Corollary 3.3.** *Let $m \in \mathbb{Z}^+$ and $l \in \mathbb{R}$. If $g \in L^2(\mathbb{R})$ and $f \in L^p(\mathbb{R})$, $p \in [2,\infty]$, with*

$$\text{distance}(\text{supp}(f);\text{supp}(g)) \geq \delta > 0,$$

then

$$\|f\,\partial_x^m J^l g\|_2 \leq c\|f\|_p\|g\|_2. \tag{3.8}$$ abc

Next, let $\theta_j \in C^\infty(\mathbb{R}) - \{0\}$ with $\theta_j' \in C_0^\infty(\mathbb{R})$ for $j = 1,2$ and

$$\text{distance}(\text{supp}(1-\theta_1);\text{supp}(\theta_2)) \geq \delta > 0. \tag{3.9}$$ teta

L1 **Lemma 3.4.** *Let $f \in H^s(\mathbb{R})$, $s < 0$, and T_a be a pseudo-differential operator of order zero ($a \in S^0$). If $\theta_1 f \in L^2(\mathbb{R})$, then*

$$\theta_2 T_a f \in L^2(\mathbb{R}). \tag{3.10}$$ 11

The next lemma is essential in the proof of Theorem 3.1.

pseudo-l2 **Lemma 3.5.** *Let $f \in L^2(\mathbb{R})$ and*

$$J^s f = (1-\partial_x^2)^{s/2} f \in L^2(\{x > a\}) \qquad s > 0,$$

then for any $\varepsilon > 0$ and any $r \in (0,s]$

$$J^r f \in L^2(\{x > a+\varepsilon\}). \tag{3.11}$$ inter

Quasilinear KdV type equations.

The aim of this sub-section is to study special regularity properties of solutions to the initial value problem (IVP) associated to some nonlinear dispersive equations of KdV type.

The result in view shows that the regularity phenomena described above is in fact more general. In particular, it is valid for solutions of the general quasilinear equation of KdV type, that is,

$$\begin{cases} \partial_t u + a(u,\partial_x u,\partial_x^2 u)\,\partial_x^3 u + b(u,\partial_x u,\partial_x^2 u) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{3.12}$$ ql-kdv

where the functions $a, b : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ satisfy:

(H1) $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ are C^∞ with all derivatives bounded in $[-M,M]^3$, for any $M > 0$,

(H2) given $M > 0$, there exists $\kappa > 0$ such that

$$1/\kappa \leq a(x,y,z) \leq \kappa \quad \text{for any} \quad (x,y,z) \in [-M,M]^3,$$

and

$$\partial_z b(x,y,z) \leq 0 \quad \text{for} \quad (x,y,z) \in [-M,M]^3.$$

To establish the propagation of regularity in solutions of (3.12) of the kind described in Theorem (2.2) we shall follow the arguments and results obtained in [20].

As before we first describe the class of solutions for which the regularity properties applies. Assuming the hypotheses (H1) and (H2), the local existence and uniqueness result established in [20] affirms:

thm8-a **Theorem 3.6** ([20]). *Let $m \in \mathbb{Z}^+$, $m \geq 7$. For any $u_0 \in H^m(\mathbb{R})$, there exist $T = T(\|u_0\|_{7,2}) > 0$ and a unique solution $u = u(x,t)$ of the IVP* (3.12) *satisfying,*

$$u \in L^\infty([0,T]; H^m(\mathbb{R})).$$

Moreover, for any $R > 0$

$$\int_0^T \int_{-R}^R (\partial_x^{m+1} u)^2(x,t)\, dxdt < \infty.$$

This result is not enough to our purpose. In the current case we need some (weak) continuous dependence of the solutions upon the data. Hence, we employed the following "refinement" of Theorem 3.6 proved in [81].

thm8-b **Theorem 3.7** ([81]). *Let $m \in \mathbb{Z}^+$, $m \geq 7$. For any $u_0 \in H^m(\mathbb{R})$ there exist $T = T(\|u_0\|_{7,2}) > 0$ and a unique solution $u = u(x,t)$ of the IVP* (3.12) *such that*

$$u \in C([0,T] : H^{m-\delta}(\mathbb{R})) \cap L^\infty([0,T] : H^m(\mathbb{R})), \text{ for all } \delta > 0, \tag{3.13}$$ i3

with

$$\partial_x^{m+1} u \in L^2([0,T] \times [-R,R]), \text{ for all } R > 0. \tag{3.14}$$ i4

Moreover, the map data solution $u_0 \mapsto u(\cdot,t)$ is locally continuous from $H^m(\mathbb{R})$ into $C([0,T] : H^{m-\delta}(\mathbb{R}))$ for any $\delta > 0$.

The main result regarding the solutions of the IVP (3.12) is the following:

thm8 **Theorem 3.8** ([81]). *Let $n, m \in \mathbb{Z}^+$, $n > m \geq 7$. If $u_0 \in H^m(\mathbb{R})$ and for some $x_0 \in \mathbb{R}$*

$$\partial_x^j u_0 \in L^2((x_0,\infty)) \text{ for } j = m+1, \dots, n,$$

then the solution of the IVP (3.12) *provided by Theorem 3.7 satisfies that for any $\varepsilon > 0$, $v > 0$, and $t \in [0,T)$*

$$\begin{aligned} &\int_{x_0+\varepsilon-vt}^{\infty} |\partial_x^j u(x,t)|^2\, dx \\ &\qquad \leq c(\varepsilon; v; \|u_0\|_{m,2}; \|\partial_x^l u_0\|_{L^2((x_0,\infty))} : l = m+1, \dots, n), \end{aligned} \tag{3.15}$$ thm2a

for $j = m+1, \dots, n$.

Moreover, for any $\varepsilon > 0$, $v > 0$, *and* $R > 0$

$$\begin{aligned}&\int_0^T \int_{x_0+\varepsilon-vt}^{x_0+R+vt} |\partial_x^{n+1} u(x,t)|^2\,dxdt \\ &\qquad \leq c(\varepsilon; v; R; \|u_0\|_{m,2}; \|\partial_x^l u_0\|_{L^2((x_0,\infty))} : l = m+1,\dots,n).\end{aligned} \tag{3.16}$$

thm2b

Several direct consequences can be deduced from Theorem 3.8 for instance. Further implications can be seen in Section 2.

Corollary 3.9. *Let* $u \in C([0,T] : H^m(\mathbb{R}))$, $m \geq 7$, *be the solution of the IVP* (3.12) *provided by Theorem 1.1. If there exist* $n > m$, $a \in \mathbb{R}$ *and* $\hat{t} \in (0,T)$ *such that*

$$\partial_x^n u(\cdot,\hat{t}) \notin L^2((a,\infty)),$$

then for any $t \in (0,\hat{t})$ *and any* $\beta \in \mathbb{R}$

$$\partial_x^n u(\cdot,t) \notin L^2((\beta,\infty)).$$

Remark 3.10. *Theorem 3.8 affirms that the propagation phenomenon described in Theorem 2.2 still holds in solutions of the quasilinear problem* (3.12). *This result and those described for KdV and other nonlinear dispersive equation seem to indicate that the propagation of regularity phenomena can be established in systems where Kato's smoothing effect (*[55]*) can be proved by integration by parts directly in the differential equation, see also* [71].

Since the arguments follow closely those in [20] without lost of generality the proofs of Theorem 3.8 and Theorem 3.7 can be restricted to the case of the model equation

$$\partial_t u + (1 + (\partial_x^2 u)^2)\partial_x^3 u = 0. \tag{3.17}$$

quasi

For the details see [81].

4. NONLOCAL ONE DIMENSIONAL MODELS

In this section we will comment on propagation of regularity properties for nonlocal nonlinear one dimensional dispersive models. These include the Benjamin-Ono equation, the Intermediate Long Wave equation and the dispersion generalized Benjamin-Ono and describe the regularity properties satisfied by solutions to the corresponding IVP.

Benjamin-Ono equation. We first consider solutions to the IVP associated to the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u - \partial_x^2 \mathscr{H} u + u\,\partial_x u = 0, \quad x,t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

bo1

where $\mathscr{H}$ denotes the Hilbert transform,

$$\begin{aligned} \mathscr{H} f(x) &= \frac{1}{\pi}\text{p.v.}\big(\frac{1}{x} * f\big)(x) \\ &= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0} \int\limits_{|y|\geq\varepsilon} \frac{f(x-y)}{y}\,dy = (-i\,\text{sgn}(\xi)\widehat{f}(\xi))^{\vee}(x). \end{aligned} \tag{4.2}$$

hita

The BO equation was first deduced by Benjamin [4] and Ono [92] as a model for long internal gravity waves in deep stratified fluids. Later, it was also shown to be a completely integrable system (see [2], [17] and references therein).

The problem of finding the minimal regularity property, measured in the Sobolev scale

$$H^s(\mathbb{R}) = \big(1-\partial_x^2\big)^{-s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R},$$

required to guarantee that the IVP (4.1) is locally or globally well-posed in $H^s(\mathbb{R})$ has been extensively studied. Thus, one has the following list of works [1], [44], [94], [67], [58], [106], [11] and [43] where GWP was established in $H^0(\mathbb{R}) = L^2(\mathbb{R})$, (for further details and results regarding the well-posedness of the IVP (4.1) in $H^s(\mathbb{R})$ we refer to [86]). It should be pointed out that a result found in [87] (see also [68]) implies that none well-posedness in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$ for the IVP (4.1) can be established by a solely contraction principle argument.

Well-posedness of the IVP (4.1) has been also studied in weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \qquad s,\, r \in \mathbb{R}, \tag{4.3}$$

spaceZ

and

$$\dot{Z}_{s,r} = \{f \in H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx) \,:\, \widehat{f}(0) = 0\}, \qquad s,\, r \in \mathbb{R}. \tag{4.4}$$

spaceZdot

To state the regularity property for solutions of the BO equation we shall use the family of functions $\chi_{0,\varepsilon,b} \in C^{\infty}(\mathbb{R})$ introduced in (2.14)-(2.25).

pro-BO

Theorem 4.1 ([48])**.** *Let $u_0 \in H^{3/2}(\mathbb{R})$ and $u = u(x,t)$ be the corresponding solution of the IVP* (4.1) *provided by Theorem A. If for some $x_0 \in \mathbb{R}$ and for some $m \in \mathbb{Z}^+$, $m \geq 2$,*

$$\int_{x_0}^{\infty} (\partial_x^m u_0)^2(x)\,dx < \infty, \tag{4.5}$$

bo3

then for any $v>0$, $T>0$, $\varepsilon>0$, $b\geq 5\varepsilon$

$$\begin{aligned}&\sup_{0\leq t\leq T}\int(\partial_x^m u(x,t))^2\,\chi_{0,\varepsilon,b}(x-x_0+vt)\,dx\\&+\int_0^T\int(D_x^{1/2}\partial_x^m u(x,t))^2\chi'_{0,\varepsilon,b}(x-x_0+vt)\,dxdt<c=c(T,\varepsilon,b,v).\end{aligned}\tag{4.6}$$ bo4

If in addition to (4.5) *there exists $x_0\in\mathbb{R}$ such that any $\varepsilon>0$, $b>5\varepsilon$*

$$D_x^{1/2}(\partial_x^m u_0\,\chi_{0,\varepsilon,b}(\cdot-x_0))\in L^2(\mathbb{R}),\tag{4.7}$$ bo5

then

$$\begin{aligned}&\sup_{0\leq t\leq T}\int(D_x^{1/2}(\partial_x^m u(x,t)\,\chi_{0,\varepsilon,b}(x-x_0+vt)))^2\,dx\\&+\int_0^T\int(\partial_x^{m+1}u(x,t))^2\,\chi'_{0,\varepsilon,b}\,\chi_{0,\varepsilon,b}(x-x_0+vt)\,dxdt<c,\end{aligned}\tag{4.8}$$ bo6

with $c=c(T,\varepsilon,b,v)$.

Remark 4.2. *The results in Theorem 4.1 can be extended to solutions of the IVP associated to the k-generalized BO equation*

$$\partial_t u-H\partial_x^2 u+u^k\,\partial_x u=0,\quad x,t\in\mathbb{R},\quad k\in\mathbb{Z}^+.$$

The following are direct implications of the above comment, the group properties, and Theorems 4.1 . In order to simplify the exposition we shall state them only for solutions of the IVP (4.1). First, as a direct consequence of Theorem 4.1 and the time reversible character of the equation in (4.1) one has:

cor.1 **Corollary 4.3.** *Let $u\in C(\mathbb{R}:H^{3/2^+}(\mathbb{R}))$ be a solution of the equation in* (4.1) *described in Theorem A. If there exist $m\in\mathbb{Z}^+$, $m\geq 2$, $\hat{t}\in\mathbb{R}$, $a\in\mathbb{R}$ such that*

$$\partial_x^m u(\cdot,\hat{t})\notin L^2((a,\infty)),$$

then for any $t\in(-\infty,\hat{t})$ and any $\beta\in\mathbb{R}$

$$\partial_x^m u(\cdot,t)\notin L^2((\beta,\infty)).$$

Next, one has that for appropriate class of data singularities of the corresponding solutions travel with infinite speed to the left as time evolves.

cor.2 **Corollary 4.4.** *Let $u \in C(\mathbb{R} : H^{3/2}(\mathbb{R}))$ be a solution of the equation in* (4.1) *described in Theorem A. If there exist $k, m \in \mathbb{Z}^+$ with $k \geq m$ and $a, b \in \mathbb{R}$ with $b < a$ such that*

$$\int_a^{\infty} |\partial_x^k u_0(x)|^2 dx < \infty \quad but \quad \partial_x^m u_0 \notin L^2((b,\infty)), \tag{4.9}$$ AA

then for any $t \in (0,\infty)$ and any $v > 0$ and $\varepsilon > 0$

$$\int_{a+\varepsilon-vt}^{\infty} |\partial_x^k u(x,t)|^2 dx < \infty,$$

and for any $t \in (-\infty, 0)$ and $\alpha \in \mathbb{R}$

$$\int_{\alpha}^{\infty} |\partial_x^m u(x,t)|^2 dx = \infty.$$

Remark 4.5. *(1) In comparison to the proof of the propagation of regularity for solutions of he KdV equation, the proof for the BO equation considered here is quite more involved. First, it includes a non-local operator, the Hilbert transform* (4.2)*. Second, in the case of the k-gKdV the local smoothing effect yields a gain of one derivative which allows to pass to the next step in the inductive process. However, in the case of the BO equation the gain of the local smoothing is just* $1/2$*-derivative so the iterative argument has to be carried out in two steps, one for positive integers m and another one for* $m + 1/2$*.*

(2) In addition, the explicit identity (1.4) *describing the local smoothing effect in solutions of the KdV equation is not available for the BO equation. In this case, to establish the local smoothing one has to rely on several commutator estimates. The main one is the extension of the Calderón first commutator estimate for the Hilbert transform* [14] *given in* [3]*. More precisely,*

Cal **Theorem 4.6** ([3])**.** *Let $\mathcal{H}$ be the Hilbert transform. For any $p \in (1,\infty)$ and any $l, m \in \mathbb{Z}^+$, $l + m \geq 1$ there exists $c = c(p; l; m) > 0$ such that*

$$\|\partial_x^l [\mathcal{H}; \psi]\, \partial_x^m f\|_p \leq c\, \|\partial_x^{m+l} \psi\|_\infty \|f\|_p. \tag{4.10}$$ CE

For a different proof of this estimate see Lemma 3.1 in [21]*.*

Intermediate Long Wave equation. The next nonlocal model in consideration is the Intermedite Long Wave (ILW) equation,

$$\partial_t u + \mathcal{T}_\delta \partial_x^2 u + \frac{1}{\delta} \partial_x u + u \partial_x u = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R},\ \delta > 0, \tag{4.11}$$ ilw

where $u = u(x,t)$ is a real value function and

$$\mathcal{T}_\delta f(x) := -\frac{1}{2\delta} \mathrm{p.v.} \int \coth\Big(\frac{\pi(x-y)}{2\delta}\Big) f(y)\, dy.$$

Observe that $\mathscr{T}_\delta$ is an order zero Fourier multiplier, in the sense that $\partial_x \mathscr{T}_\delta$ is the multiplier with symbol

$$\sigma(\partial_x \mathscr{T}_\delta) = \widehat{\partial_x \mathscr{T}_\delta} = -2\pi\xi \coth(2\pi\delta\xi).$$

The above describes the long internal gravity waves in a two layers stratified fluid having a large finite depth represented by the parameter δ see [52, 53, 72, 15, 98].

In [1] it was shown that solutions of the ILW as $\delta \to \infty$ (infinite depth limit) converge to solutions of the BO equation (4.1) and also that if $u_\delta(x,t)$ denotes the solution of the ILW equation (4.11), then

$$v_\delta(x,t) = \frac{3}{\delta} u_\delta\big(x, \frac{3}{\delta} t\big)$$

converge as $\delta \to 0$ (shallow-water limit) to the KdV equation (1.2) ($k = 1$) with the same initial data.

The ILW have been proven to be completely integrable see [69], [70]. For further comments on general properties of the ILW we refer to [99] and references therein.

pr-ilw **Theorem 4.7** ([88]). *Let $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and*

$$u \in C(\mathbb{R} : H^s(\mathbb{R})) \cap C^1([0,T] : H^{s-2}(\mathbb{R})) \cap L^\infty([0,T] : H^{s+1/2}_{\text{loc}}(\mathbb{R})),$$

be the corresponding solution of the IVP associated to the ILW equation (4.11) *with data u_0 provided by the local theory established in Theorem 6.1 in* [1]. *If for some $x_0 \in \mathbb{R}$, $m \in \mathbb{Z}^+$ and $m > s$ one has that*

$$M_0 := \int_{x_0}^{\infty} (\partial_x^m u_0(x))^2 \, dx < \infty, \tag{4.12}$$ ilw-1

then for any $\gamma > 0$, $T > 0$, $\varepsilon > 0$, $R > 5\varepsilon$ it follows that

$$\begin{aligned} \sup_{0 \le t \le T} \int_{x_0+\varepsilon-\gamma t} (\partial_x^m(x,t))^2 \, dx + \int_0^T \int_{x_0+\varepsilon-\gamma t}^{x_0+R-\gamma t} (D^{1/2} \partial_x^m(x,t))^2 \, dx \\ \le c = c(\|u_0\|_{s,2}; T; M_0; \gamma, \varepsilon, R). \end{aligned} \tag{4.13}$$ ilw-2

The scheme to prove Theorem 4.7 was reduced to the proof provided for solutions of the BO equation in [48]. We shall sketch that idea here:

5. Proof of Theorem 4.7

Proof. Consider the principal symbol of the operator modeling the dispersive relation in the ILW equation (4.11)

$$\sigma(\mathscr{T}_\delta \, \partial_x^2) = -4\pi^2 \xi^2 \coth(2\pi\delta\xi) \, i. \tag{5.1}$$ 3.1

Thus,

$$\begin{aligned}
&4\pi^2|\xi|\xi - 4\pi^2\xi^2\coth(2\pi\delta\xi)\\
&= 4\pi^2|\xi|\xi\Big(1-\operatorname{sgn}(\xi)\coth(2\pi\delta\xi)\Big)\\
&= 4\pi^2|\xi|\xi\Big(-\frac{2e^{-4\pi\delta|\xi|}}{1-e^{-4\pi\delta|\xi|}}\Big).
\end{aligned} \tag{5.2}$$

3.2

Hence, for any $M>0$ sufficiently large (such that $e^{-4\pi^2\delta M}\le 1/2$) one has

$$\sup_{|\xi|\ge M} ||4\pi^2|\xi|\xi - 4\pi^2\xi^2\coth(2\pi\delta\xi)| \le 16M^2 e^{-4\pi^2\delta M}. \tag{5.3}$$

3.3

Now, taking $\chi\in C_0^\infty(\mathbb{R})$ such that supp $\subset[-1,1]$ with $\chi(x)=1, x\in[-1/2,1/2]$ and $\chi(x)\ge 0$, $x\in\mathbb{R}$, one sees from (5.3) that for any $R>0$

$$\sigma_{R,1}(\xi) = (4\pi^2|\xi|\xi - 4\pi^2\xi^2\coth(2\pi\delta\xi))\Big(1-\chi(\xi/R)\Big)\in S^{-\infty}, \tag{5.4}$$

symbl1

i.e. $\sigma_{R,1}$ is the symbol of a smoothing pseudo-differential operator and that for any $R>0$

$$\sigma_{R,2}(\xi) = (4\pi^2|\xi|\xi - 4\pi^2\xi^2\coth(2\pi\delta\xi))\,\chi(x/R), \tag{5.5}$$

symbl2

is a multiplier with compact support.

Hence, the operator $A_{\delta_R}(\partial_x)$ with symbol $\sigma(A_{\delta_R})(\xi) = 4\pi^2|\xi|\xi - 4\pi^2\xi^2\coth(2\pi\delta\xi)$ satisfies that for any $s\ge 0$

$$\|A_{\sigma_R}f\|_{s,2}\le c\,\|f\|_2, \quad \text{with} \quad c=c(s;\delta). \tag{5.6}$$

estimate111

Therefore, by rewriting the ILW equation (4.11) as

$$\partial_t u + \mathscr{H}\,\partial_x^2 u + (\mathscr{T}_\delta\partial_x^2 - \mathscr{H}\,\partial_x^2)u + \frac{1}{\delta}\partial_x u + u\partial_x u = 0, \tag{5.7}$$

moILW

one has that the argument carried out in [48] for the BO equation based on weighted energy estimates can be applied for the equation in (5.7) without any major modification.

This essentially yields the proof of Theorem 4.7. □

Dispersion generalized Benjamin-Ono equation. To end this section we discuss the regularity properties for solutions of the IVP associated to the dispersion generalized Benjamin-Ono (dgBO) equation, namely,

$$\partial_t u + u\partial_x u - D_x^{\alpha+1}\partial_x u = 0, \;\; k\in\mathbb{Z}^+, \;\; 1\le\alpha\le 2, \tag{5.8}$$

dgbo

where D_x^s denotes the homogeneous derivative of order $s\in\mathbb{R}$,

$$D_x^s = (-\partial_x^2)^{\frac{s}{2}} \quad \text{thus} \quad \widehat{D_x^s f}(\xi) = c_s(|\xi|^s\widehat{f}(\xi)).$$

These equations model vorticity waves in the coastal zone (see [87]). We can observe that for $\alpha = 1$ corresponds to the BO equation and $\alpha = 2$ to the KdV equation. Next we present a result proved in [84] where all the cases $1 < \alpha < 2$ are treated.

pro-dgbo

Theorem 5.1 ([84]). *Let $u_0 \in H^s(\mathbb{R})$, with $s = \frac{3-\alpha}{2}$, and $u = u(x,t)$ be the corresponding solution of the IVP associated to* (5.8) *provided by the local theory established in* [84] *Theorem A.*

If for some $x_0 \in \mathbb{R}$ and for some $m \in \mathbb{Z}^+$, $m \geq 2$,

$$\partial_x^m u_0 \in L^2(\{x \geq x_0\}), \tag{5.9}$$

pro-dgbo-1

then for any $v \geq 0$, $T > 0$, $\varepsilon > 0$ and $\tau > \varepsilon$

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{x_0+\varepsilon-vt} (\partial_x^j u)^2(x,t)\,dx \\ &\quad + \int_0^T \int_{x_0+\varepsilon-vt}^{x_0+\tau-vt} (D_x^{\frac{\alpha+1}{2}} \partial_x^j u)^2(x,t) \\ &\quad + \int_0^T \int_{x_0+\varepsilon-vt}^{x_0+\tau-vt} (D_x^{\frac{\alpha+1}{2}} \mathscr{H} \partial_x^j u)^2(x,t) \leq c \end{aligned} \tag{5.10}$$

pro-dgbo-2

for $j = 1, 2, \dots, m$, with $c = c(T;\varepsilon;v;\alpha;\|u_0\|_{H^s};\|\partial_x^m u_0\|_{L^2((x_0,\infty))}) > 0$.

If in addition to (5.9) *there exists $x_0 \in \mathbb{R}^+$ with*

$$D_x^{\frac{\alpha+1}{2}} \partial_x^m u_0 \in L^2(\{x \geq x_0\}) \tag{5.11}$$

then for any $v \geq 0$, $\varepsilon > 0$ and $\tau > \varepsilon$

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{x_0+\varepsilon-vt} (D_x^{\frac{1-\alpha}{2}} \partial_x^m u)^2(x,t)\,dx \\ &\quad + \int_0^T \int_{x_0+\varepsilon-vt}^{x_0+\tau-vt} (\partial_x^{m+1} u)^2(x,t) \\ &\quad + \int_0^T \int_{x_0+\varepsilon-vt}^{x_0+\tau-vt} (\partial_x^{m+1} \mathscr{H} u)^2(x,t) \leq c \end{aligned} \tag{5.12}$$

with $c = c(T;\varepsilon;v;\alpha;\|u_0\|_{H^s}, \|D_x^{(1-\alpha)/2} \partial_x^m u_0\|_{L^2((x_0,\infty))}) > 0$.

Remark 5.2. *Although the argument of the proof of Theorem 5.1 follows in spirit that of KdV, i.e. an induction process combined with weighted energy estimates, the presence of the nonlocal operator $D_x^{\alpha+1}\partial_x$ in the term modeling the dispersion, makes the proof much harder. More precisely, two difficulties appear, the most important of which is to obtain explicitly the Kato smoothing effect, which in the proof of Theorem 2.2 is fundamental.*

In contrast to the KdV equation, the gain of the local smoothing in solutions of the dispersive generalized Benjamin-Ono equation is just $\frac{\alpha+1}{2}$

derivatives, so as occurs in the case of the Benjamin-Ono equation [48]*, the iterative argument in the induction process is carried out in two steps, one for positive integers m and another one for* $m+\frac{1-\alpha}{2}$ *derivatives*

In the case of the BO equation [48]*, the smoothing effect is obtained basing their analysis on several commutator estimates, such as the extension of Calderón's first commutator for the Hilbert transform Theorem 4.6. However, the method of proof does not provide explicitly the local smoothing as in* (1.4)*. The advantage of this method is that it allows us to obtain explicitly the smoothing effect for any* $\alpha \in (0,1)$ *in the IVP* (5.8)*. Roughly, we write the term modeling the dispersive part of the equation in* (5.8) *in terms of an expression involving the commutator* $[\mathscr{H} D_x^{\alpha+2}, \chi_{0,\varepsilon,b}^2]$.

At this point, results in [31] *(see Proposition 5.3 below) about commutator decomposition are blended. This allows to obtain explicitly the smoothing effect as in* (1.4) *at every step of the induction process in the energy estimate. Additionally, this approach allows the study of the propagation of regularity phenomena in models where the dispersion is lower in comparison with that of IVP* (5.8) *(see* [85]*).*

For completeness we include the commutator expansions derived in [30] and [31].

Let $a=2\mu+1>1$ and $n\in\mathbb{N}$. Let g be a smooth function such that its derivative has a mild decay at infinity.

We consider the operator

$$\mathscr{R}_n(a) = [HD_x^a; g] - \frac{1}{2}\left(\mathscr{P}_n(a) - H\mathscr{P}_n(a)H\right),$$

where

$$\mathscr{P}_n(a) = a\sum_{0\le j\le n} c_{2j+1}(-1)^j 4^{-j} D_x^{\mu-j}\left(g^{(2j+1)} D_x^{\mu-j}\right)$$

and

$$c_1 = 1, \quad c_{2j+1} = \frac{1}{(2j+1)!}\prod_{0\le k<j}\left(a^2-(2k+1)^2\right), \quad H = -\mathscr{H}.$$

The decomposition above was obtained by [30].

Propo2 **Proposition 5.3.** *Let n be a non-negative integer,* $a\ge 1$ *and* $\sigma\ge 0$ *be such that*

$$2n+1 \le a+2\sigma \le 2n+3. \tag{5.13}$$

Then

(a) *The operator* $D_x^{\sigma}\mathscr{R}_n(a)D_x^{\sigma}$ *is bounded in* $L^2(\mathbb{R})$ *with norm*

$$\|D_x^{\sigma}\mathscr{R}_n(a)D_x^{\sigma} f\|_{L^2_x} \le \frac{c}{\sqrt{2\pi}}\left\|\mathscr{F}_{\xi}\left(D_x^{a+2\sigma} g\right)\right\|_{L^1_{\xi}} \|f\|_{L^2_x}. \tag{5.14}$$ continuity

If $a \geq 2n+1$, one can take $c = 1$.

(b) *Assume in addition that*

$$2n+1 \leq a+2\sigma < 2n+3.$$

Then the operator $D_x^{\sigma}\mathscr{R}_n D_x^{\sigma}$ is compact in $L^2(\mathbb{R})$.

Proof. See Proposition 2.2 in [31]. □

6. Higher Dimensional Models

In this section we will comment on propagation of regularity for solutions of Kadomtsev-Petviashvili II (KPII) equation and the Zakharov-Kuznetsov (ZK) equation.

KPII equation.

In [49] it was considered the propagation of regularity for solutions of the IVP associated to the KPII equation,

$$\partial_t u + \partial_x^3 u + \alpha\, \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \quad (x,y) \in \mathbb{R}^2, t \in \mathbb{R},\ \alpha = 1, \tag{6.1}$$

KPII

where the nonlocal operator ∂_x^{-1} is defined via Fourier transform as

$$\widehat{\partial_x^{-1} f}(\xi) = \frac{1}{i\xi}\widehat{f}(\xi).$$

The KP equations (KPI ($\alpha = -1$) and KPII ($\alpha = 1$)) are models for the propagation of long, dispersive, weakly nonlinear waves which travel predominantly in the x direction, with weak transverse effects. These equations were derived in [54] as two-dimensional extensions of the KdV equation (1.2). The KP equations are completely integrable systems and have been studied extensively in the last few years in several aspects. For an interesting account of KP equations features and open problems we refer the reader to [66].

Concerning well-posedness in low regularity spaces for the IVP associated to the KP equations we shall give an incomplete list of the most significative results in this aspect, [9], [105], [50], [103], [109], [110], [104], [40], [41], [51].

To state the propagation of regularity properties for solutions of the IVP associated to the KPII equation we introduce the following notation.

$$X_s = \{f \in H^s(\mathbb{R}^2) \,:\, \partial_x^{-1} f \in H^s(\mathbb{R}^2)\}. \tag{6.2}$$

XS

thm9-kp

Theorem 6.1. *Let u be a solution of the IVP associated to* (6.1) *corresponding to initial data $u_0 \in X_s(\mathbb{R}^2)$, $s > 2$ provided in Theorem 2.2 in* [45]. *Suppose that for an integer $n \geq 3$ and some $x_0 \in \mathbb{R}$, the restriction of u_0 to $(x_0,\infty) \times \mathbb{R}$ belongs to $H^n((x_0,\infty)\times\mathbb{R})$ and $\partial_x^{-1}\partial_y^3 u_0 \in L^2((x_0,\infty)\times\mathbb{R})$.*

Then, for any $\nu > 0$ *and* $\varepsilon > 0$

$$\sup_{t\in[0,T]} \sum_{\alpha_1+\alpha_2\le n} \int_{-\infty}^{\infty} \int_{x_0+\varepsilon-\nu t}^{\infty} (\partial_x^{\alpha_1}\partial_y^{\alpha_2} u(x,y,t))^2\, dx\, dy < \infty. \tag{6.3}$$

main

In particular, for all times $t \in (0,T]$ *and for all* $a \in \mathbb{R}$, $u(t) \in H^n((a,\infty)\times\mathbb{R})$.

The proof uses similar argument as those given to obtain the result for the KdV case in Theorem 2.2. We shall notice that solutions of the IVP associated to the KPII equation also share a smoothing property similar to the one in (1.4) established in [55] for solutions of the Korteweg-de Vries equation.

ZK equation.

Next we will focus on the smoothing properties for solutions ZK equation for which the propagation of regularity occurs in "cones".

We consider solutions of the initial value problem (IVP) associated to the three dimensional (3D) Zakharov-Kuznetsov (ZK) equation

$$\begin{cases} \partial_t u + \partial_x \Delta u + u\partial_x u = 0, \quad (x,y,z) \in \mathbb{R}^3, t \in \mathbb{R}, \\ u(\vec{x},y,z,0) = u_0(x,y,z) \end{cases} \tag{6.4}$$

zk-3d

where u is a real function and Δ denotes the Laplace operator in space variables.

The equation above arises in the context of plasma physics, it was formally derived in [111] as a long wave small-amplitude limit of the Euler-Poisson system in the "cold plasma" approximation. This formal long-wave limit was rigorously justified in [73]. This model is still valid in two dimensions.

The ZK equation can be seen as a natural multi-dimensional extension of the KdV equation, quite different from the above mentioned KPII equation which is obtained as an asymptotic model of various nonlinear dispersive systems under a different scaling. Contrary to the KdV or the KP equations, the ZK equation is not completely integrable.

Regarding well-posedness for the IVP associated to the equation in (6.4) as well as its generalizations in dimension $d \ge 2$ we refer to [95] , [86], [78], [86], [35], [24], [74], [96], [34], [75], [33], [96], [34], [75], [33]. For further issues related to the ZK equation we refer to [8], [19], [25], [93], [12], [13], [5], and [77].

As we have seen along these notes the property described in Theorem 2.2 is intrinsic to suitable solutions of many nonlinear dispersive models.

As above we first describe the class of solutions in which it applies. Thus, we first recall a result which is direct consequence of the energy estimates obtained by combining the commutator estimates in [56], the Sobolev embedding theorem and the argument in [7].

ZK-3D **Theorem 6.2.** *Given $u_0 \in H^s(\mathbb{R}^3)$ with $s > 5/2$, there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(\vec{x},t)$ of the IVP* (6.4) *such that*

$$u \in C([0,T] : H^s(\mathbb{R}^3)). \tag{6.5}$$ A1

Moreover, the map data-solution $u_0 \mapsto u(\vec{x},t)$ from $H^s(\mathbb{R}^3)$ into $C([0,T] : H^s(\mathbb{R}^3))$ is (locally) continuous.

We observe that

$$\partial_x u, \partial_y u, \partial_z u \in C([0,T] : H^{s-1}(\mathbb{R}^3)) \subset L^1([0,T] : L^\infty(\mathbb{R}^3)). \tag{6.6}$$ A2

Also, we shall use the following result which is a consequence of the arguments given in [78].

ZK-3Db **Theorem 6.3.** *Given $u_0 \in H^s(\mathbb{R}^3)$ with $s > 3/2$, there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(\vec{x},t)$ of the IVP* (6.4) *such that*

$$u \in C([0,T] : H^s(\mathbb{R}^3)),$$

with

$$\partial_x u \in L^1([0,T] : L^\infty(\mathbb{R}^3)).$$

Moreover, the map data-solution $u_0 \mapsto u(\vec{x},t)$ from $H^s(\mathbb{R}^3)$ into $C([0,T] : H^s(\mathbb{R}^3))$ is (locally) continuous.

To delineate our results we introduce some notations: for $a, b, c, d, f \in \mathbb{R}$ we define the half-space

$$P_{\{a,b,c,d\}} = \{(x,y,z) \in \mathbb{R}^3 \,:\, ax + by + cz \geq d\},$$

and the strip

$$H_{\{a,b,c,d,f\}} = \{(x,y,z) \in \mathbb{R}^3 \,:\, d \leq ax + by + cz \leq f\}.$$

Our main result in this section is the following :

thm9 **Theorem 6.4.** *Let $u_0 \in H^s(\mathbb{R}^3)$ with $s > 5/2$. If for some $(a,b,c) \in \mathbb{R}^3$ with*

$$a > 0, \;\; b, c \geq 0 \quad \textit{and} \quad \sqrt{3}a > \sqrt{b^2 + c^2}, \tag{6.7}$$ main-hyp

and for some $j \in \mathbb{Z}^+,\; j \geq 3$

$$N_j \equiv \sum_{|\alpha| = j} \int_{P_{\{a,b,c,d\}}} (\partial^\alpha u_0(\vec{x}))^2 \, d\vec{x} < \infty, \tag{6.8}$$ mainhyp

then the corresponding solution $u = u(\vec{x},t)$ of the IVP for the ZK equation (6.4) *provided by Theorem 6.2 satisfies that for any $v \geq 0$, $\varepsilon > 0$ and $R > 4\varepsilon$*

$$\begin{aligned} &\sup_{0\leq t\leq T} \sum_{|\alpha|\leq j} \int_{P_{\{a,b,c,d-vt+\varepsilon\}}} (\partial^\alpha u(\vec{x},t))^2 d\vec{x} \\ &+ \sum_{|\alpha|=j+1} \int_0^T \int_{H_{\{a,b,c,d-vt+\varepsilon,d-vt+R\}}} (\partial^\alpha u(\vec{x},t))^2 d\vec{x}dt \\ &\leq c = c(\|u_0\|_{s,2}; \{N_l : 1\leq l\leq j\}; j; a; b; c; v; T; \varepsilon; R). \end{aligned} \tag{6.9}$$ main-result

R1 **Remark 6.5.** *It is clear that if the hypothesis* (6.8) *is satisfied by a finite family of half-spaces $P_{\{a_r,b_r,c_r,d_r\}}$, $r = 1,..,N$ with the (a_r,b_r,c_r)'s as in* (6.7)*, then the result in* (6.9) *holds for the union of the $P_{\{a_r,b_r,c_r,d_r-vt+\varepsilon\}}$'s and $H_{\{a_r,b_r,c_r,d_r-vt+\varepsilon,d_r-vt+R\}}$'s.*

In particular, this tells us that the result extends to sets which are complement of a class of cones. More precisely, define the solid cone with vertex $\vec{x}_0 = (x_0,y_0,z_0) \in \mathbb{R}^3$, axis $\hat{w} \in \mathbb{S}^2$ and opening $\theta \in (0,\pi)$ as

$$\mathscr{C}_{\vec{x}_0,\hat{w},\theta} = \{\vec{x} = (x,y,z) \in \mathbb{R}^3 : \langle \vec{x}-\vec{x}_0, \hat{w}\rangle \leq \|\vec{x}-\vec{x}_0\| \cos(\theta)\}. \tag{6.10}$$ cone1

Thus, if for some $j \in \mathbb{Z}^+$, $j \geq 2$

$$\mathscr{N}_j \equiv \sum_{|\alpha|=j} \int_{(\mathscr{C}_{\vec{x}_0,-e_1,\pi/6})^c} (\partial^\alpha u_0(\vec{x}))^2 d\vec{x} < \infty, \tag{6.11}$$ cone2

then for any $v \geq 0$ and $\varepsilon > 0$

$$\sup_{0\leq t\leq T} \sum_{|\alpha|\leq j} \int_{(\mathscr{C}_{\vec{x}_0-vte_1+\varepsilon e_1,-e_1,\pi/6})^c} (\partial^\alpha u(\vec{x},t))^2 d\vec{x} < c \tag{6.12}$$ cone3

with $c = c(\|u_0\|_{s,2}; \{\mathscr{N}_l : 1 \leq l \leq j\}; j; T; \varepsilon; v)$.

A similar result to that described in (6.10)-(6.12) holds in $(\mathscr{C}_{\vec{x}_0,\hat{w},\theta})^c$ (see (6.10)) assuming that

$$\mathscr{C}_{\vec{x}_0,-e_1,\pi/6} \subseteq \mathscr{C}_{\vec{x}_0,\hat{w},\theta}. \tag{6.13}$$ cone-cond

One notices that the fact $\varepsilon > 0$ allows us to write $(\mathscr{C}_{\vec{x}_0-vt\hat{v}+\varepsilon\hat{w},\hat{w},\theta})^c$ as the union of finitely many half-spaces $P_{a_r,b_r,c_r,d_r-vt-\varepsilon/2}$'s, and that the condition (6.13) guarantees that the a_r,b_r,c_r's satisfy (6.7).

The same argument given below for the proof of Theorem 6.2 shows that :

3D-regularity-b **Theorem 6.6.** *Let $u_0 \in H^s(\mathbb{R}^3)$ with $s > 3/2$. If for some $(a,b,c) \in \mathbb{R}^3$ with*

$$a > 0, \quad b,c \geq 0 \quad \textit{and} \quad \sqrt{3}a > \sqrt{b^2+c^2},$$

and for some $j \in \mathbb{Z}^+,\ j \geq 2$

$$\widetilde{N_j} \equiv \int_{P_{\{a,b,c,d\}}} (\partial_x^j u_0(\vec{x}))^2 d\vec{x} < \infty,$$

then the corresponding solution $u = u(\vec{x},t)$ *of the IVP for the ZK equation* (6.4) *provided by Theorem 6.3 satisfies that for any* $v \geq 0$, $\varepsilon > 0$ *and* $R > 4\varepsilon$

$$\begin{aligned}
&\sup_{0\leq t\leq T} \int_{P_{\{a,b,c,d-vt+\varepsilon\}}} (\partial_x^j u(\vec{x},t))^2 d\vec{x} \\
&+ \int_0^T \int_{H_{\{a,b,c,d-vt+\varepsilon,d-vt+R\}}} (\partial_x^{j+1} u(\vec{x},t))^2 d\vec{x}dt \\
&\leq c = c(\|u_0\|_{s,2}; \{N_l : 1 \leq l \leq j\}; j; a; b; c; v; T; \varepsilon; R).
\end{aligned}$$

R2b **Remark 6.7.** *The same comments stated in Remark 6.5 apply to the results in Theorem 7.3 with* ∂_x *instead of* ∂ *in* (6.11) *and* (6.12).

I like the next remark but I do not know if it is appropriated for these notes

R2 **Remark 6.8.** *Concerning the sectorial regularity of solutions of the IVP* (6.4) *we consider the so called Strichartz estimates for the linear IVP associated to* (6.4)

$$\begin{cases} \partial_t v + \partial_x \Delta v = 0, \quad (x,y,z) \in \mathbb{R}^3, t \in \mathbb{R}, \\ v(x,y,z,0) = v_0(x,y,z). \end{cases} \tag{6.14}$$ l-zk3

The solution $v(x,y,z,t)$ *of* (6.14) *is described by the unitary group* $\{U(t) : t \in \mathbb{R}\}$ *with*

$$\widehat{U(t)v_0}(\vec{\xi}\,) = e^{it\varphi(\vec{\xi})}\widehat{v}_0(\vec{\xi}\,), \quad \varphi(\vec{\xi}\,) = \xi_1^3 + \xi_1(\xi_2^2 + \xi_3^2)$$

with $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$.

In [78] *(Proposition 1) Linares-Saut proved : Let* $\varepsilon \in (0,1)$ *and* $\theta \in (0,(1+\varepsilon/3)^{-1})$*, Then*

$$\|D_x^{\theta\varepsilon/2} U(t)v_0\|_{L_t^q L_{xyz}^p} \leq c\|v_0\|_{L_{xyz}^2},$$

where $1/q + 1/q' = 1/p + 1/p' = 1$, $p = 1/(1-\theta)$ *and* $2/q = \theta(1+\varepsilon/3)$.

Thus, for $\varepsilon \sim 1^-$ *one gets a gain of* $3/8^-$ *derivatives (in the x-variable).*

Notice that (roughly speaking) this gain of derivative in the x-variable extends to all variables (x,y,z) *if* $\widehat{v}_0$ *is supported inside the cone* $\sqrt{3}\xi_1 > \sqrt{\xi_2^2 + \xi_3^2}$ *which is similar to that described in* (6.7) *for the physical space.*

In [60] *(Theorem 3.1) Kenig-Ponce-Vega showed : for $\mu \geq 0$ define*

$$U_\mu(t)v_0(\vec{x}) = \int e^{i(t\varphi(\vec{\xi})+\vec{x}\cdot\vec{\xi})}\,|H\varphi|^{\mu/2}\,\widehat{v}_0(\vec{\xi})\phi_\varphi(\vec{\xi})d\vec{\xi},$$

with

$$H\varphi(\vec{\xi}) = det(\partial^2_{jk}\varphi) = 8\xi_1(3\xi_1^2-(\xi_2^2+\xi_3^2)),$$

(hence

$$H\varphi(\vec{\xi}) \geq \delta\,\|\vec{\xi}\,\|^3 \quad \textit{if} \quad \sqrt{3}\xi_1 > \sqrt{\xi_2^2+\xi_3^2},$$

for some $\delta > 0$) and $\phi_\varphi \in C^\infty(\mathbb{R}^3)$ supported in the cone

$$\sqrt{3}\xi_1 > \sqrt{\xi_2^2+\xi_3^2}$$

with $\phi_\varphi(\vec{\xi}) = 1$ for $\vec{\xi}$ sufficiently large inside the cone

$$\sqrt{3}\xi_1 > (1+\varepsilon)\sqrt{\xi_2^2+\xi_3^2}, \quad \varepsilon > 0,$$

then

$$\|U_{\mu/2}(t)v_0\|_{L^q_tL^p_{xyz}} \leq c\|\,\widehat{v}_0\,\phi_\varphi\|_{L^2_{xyz}}, \tag{6.15}$$ stri-2

where

$$\mu \in [0,2/3), \quad 1/q+1/q' = 1/p+1/p' = 1, \quad \textit{and} \quad (q,p) = (4/3\theta, 2/(1-\theta)).$$

Thus, for $\mu \sim 2/3^-$ (6.15) *represents a gain of $1/2^-$ derivatives.*

The previous observation can also be applied to solutions of the linear problem associated to the ZK equation in 2D. More precisely, it was proved in [74] *that solutions of the linear problem, $\widehat{U(t)f} = e^{it(\xi(\xi^2+\eta^2))}\widehat{f}$ satisfy the Strichartz estimates.*

Lemma 6.9. *Let $0 \leq \varepsilon < 1/2$ and $0 \leq \theta \leq 1$. Then,*

$$\|D_x^{\theta\varepsilon/2}U(t)f\|_{L^q_tL^p_{xy}} \lesssim \|f\|_{L^2_{xy}},$$

where $p = \frac{2}{1-\theta}$ and $\frac{2}{q} = \frac{\theta(2+\varepsilon)}{3}$.

7. Additional Comments

In this section we will discuss some points regarding propagation of regularity, decay and some open problems.

We start by considering the question of the propagation of other type of regularities besides those proved before i.e. for $u_0 \in H^n((x_0,\infty))$ for some $x_0 \in \mathbb{R}$.

We recall that the next result can be obtained as a consequence of the argument given by Bona and Saut in [6].

thm10 **Theorem 7.1.** *Let $k \in \mathbb{Z}^+$. There exists*

$$u_0 \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

with $\|u_0\|_{1,2} \ll 1$ so that the solution $u(\cdot,t)$ of the IVP (1.2) *is global in time if $k \geq 4$ with $u \in C(\mathbb{R} : H^1(\mathbb{R})) \cap \dots$ and satisfies*

$$\begin{cases} u(\cdot,t) \in C^1(\mathbb{R}), \;\; t > 0, \quad t \notin \mathbb{Z}^+, \\ u(\cdot,t) \in C^1(\mathbb{R}\backslash\{0\}) \backslash C^1(\mathbb{R}), \quad t \in \mathbb{Z}^+. \end{cases} \tag{7.1}$$ i10

The argument in [6] is based in a careful analysis of the asymptotic decay of the Airy function and the well-posedness of the IVP (1.2) with data $u_0(x)$ in appropriate weighted Sobolev spaces. This argument was simplified (for the case of two points in (7.1)) for the modified KdV equation $k = 2$ in [78] without relying in weighted spaces. Here we shall give a direct proof of Theorem 1.5 which follows the approach in [82], i.e. it does not rely on the analysis of the decay of the Airy function and applies to all the nonlinearities.

Our method has the advantage that it can be extended to $W^{s,p}$-setting. More precisely, we shall show the following:

thm11 **Theorem 7.2.**

(a) *Fix $k = 2, 3, \dots$, let $p \in (2, \infty)$ and $j \geq 1$, $j \in \mathbb{Z}^+$. There exists*

$$u_0 \in H^{3/4}(\mathbb{R}) \cap W^{j,p}(\mathbb{R}) \tag{7.2}$$ kdv-lp1

such that the corresponding solution

$$u \in C([-T,T] : H^{3/4}(\mathbb{R})) \cap \dots$$

of (1.2) *satisfies that there exists $t \in [0,T]$ such that*

$$u(\cdot, \pm t) \notin W^{j,p}(\mathbb{R}^+). \tag{7.3}$$ kdv-lp2b

(b) *For $k = 1$, the same result holds for $j \geq 2$, $j \in \mathbb{Z}^+$.*

Remark. *It will follow from the proof that there exists u_0 as in* (7.2) *such that* (7.3) *holds in $\mathbb{R}^-$. Hence, one can conclude that regularities in $W^{j,p}(\mathbb{R})$ for $p > 2$ do not propagate forward or backward in time to the right or to the left.*

Regarding decay and persistence properties results as the one given obtained for solutions of the IVP associated to the KdV equation in Theorem 2.4 the ZK equation also enjoy persistence properties and regularity effects, for positive times, in solutions associated with data having polynomial decay in an appropriate half-space. More precisely, thm9-b **Theorem 7.3.** *If $u_0 \in H^s(\mathbb{R}^3)$, $s > 5/2$ and for some $j \in \mathbb{Z}^+$, $j \geq 2$ and some a, b, c satisfying* (6.7)

$$\begin{aligned}\mathscr{M}_j &= \| (ax+by+cz-d)^{j/2} u_0 \|^2_{L^2(P_{\{a,b,c,d\}})} \\ &= \int_{P_{\{a,b,c,d\}}} (ax+by+cz-d)^j \, |u_0(\vec{x})|^2 d\vec{x} < \infty,\end{aligned}$$

then the solution $u = u(\vec{x}, t)$ of the IVP (6.4) *provided by Theorem 6.2 satisfies that for any $v \geq 0$ and $\varepsilon > 0$*

$$\sup_{0 \leq t \leq T} \int_{P_{\{a,b,c,d+vt-\varepsilon\}}} |ax+by+cz-d+vt-\varepsilon|^j \, |u(\vec{x},t)|^2 d\vec{x} \leq c,$$

with $c = c(\|u_0\|_{s,2}; \{\mathscr{M}_l : 1 \leq l \leq j\}; j; a; b; c; v; \varepsilon; T)$. Moreover, for any $\varepsilon, \delta, R > 0, v \geq 0$

$$\begin{aligned}&\sup_{\delta \leq t \leq T} \sum_{|\alpha| \leq j} \int_{P_{\{a,b,c,d-vt+\varepsilon\}}} (\partial^\alpha u(\vec{x},t))^2 \, d\vec{x} \\ &\quad + \sum_{|\alpha| \leq j+1} \int_\delta^T \int_{H_{\{a,b,c,d-vt+\varepsilon,d-vt+R\}}} (\partial^\alpha u)^2(\vec{x},t) \, d\vec{x} dt \leq c,\end{aligned}$$

with $c = c(\|u_0\|_{s,2}; \{\mathscr{M}_l : 1 \leq l \leq j\}; j; a; b; c; v; \varepsilon; R; T)$.

The proof of Theorem 7.3 follows an argument similar to that given in the proof of Theorem 6.4 (for details see [47]).

R4 **Remark 7.4.** *Our argument of proof is mainly based on weighted energy estimates for which the assumption* (6.6) *is essential.*

Open questions. Consider the IVP associated to the cubic Schrödinger equation, i.e.

$$\begin{cases} i\partial_t u + \partial_x^2 u = |u|^2 u, \quad ,x,t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{7.4}$$ nls

where $u = u(x,t)$ is a complex valued.

Answer to the question (1) are unknown for solutions of the IVP (7.4).

- Propagation of regularity in fractional Sobolev spaces as in Theorem 3.1 for solutions of the KdV equation are not known so far.
- Propagation of regularity for solutions of the KdV in $H^s(\mathbb{R})$, $s \geq 0$?
- Decay and Persistence as in Theorem 2.4 for solutions to other nonlinear dispersive models.

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(F. Linares) IMPA, INSTITUTO MATEMÁTICA PURA E APLICADA, ESTRADA DONA CASTORINA 110, 22460-320, RIO DE JANEIRO, RJ, BRAZIL
Email address: linares@impa.br

(G. Ponce) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, USA.
Email address: ponce@math.ucsb.edu