Section 4.2

TOMÁS KABBABE.

- To solve an equation of the form

(1) $ay'' + by' + cy = 0$,

we make the substitution $y(t) = e^{rt}$. Hence

$$y'(t) = r \cdot e^{rt} \quad , \quad y''(t) = r^2 \cdot e^{rt} .$$

Plugging this in (1) we get.

$$a(r^2 e^{rt}) + b(re^{rt}) + c e^{rt} = 0 \Rightarrow e^{rt}(ar^2 + br + c) = 0$$

So, if we find a root r_1 of the equation $ar^2 + br + c = 0$, then $e^{r_1 t}$ will be a solution of (1). Moreover, when solving the equation $ar^2 + br + c = 0$ we have three cases:

- **Case 1**: Two different real roots r_1 and r_2
- **Case 2**: One real root r_1
- **Case 3**: Two different complex roots r_1 and r_2.

Let's study each case separately.

Case 1: If $r_1 \neq r_2$ then $e^{r_1 t}$ and $e^{r_2 t}$ are linearly independent (Remember 3C? this can be proven using the Wronskian). Hence, the general solution of (1) is

$$y(t) = c_1 \cdot e^{r_1 t} + c_2 e^{r_2 t}$$

Case 2: If r_1 is the unique solution of $ar^2 + br + c = 0$, then $e^{r_1 t}$ is clearly a solution of **(1)**. Also, $t\,e^{r_1 t}$ is a solution of (1) Too. Hence, the general solution of **(1)** will be

$$y(t) = c_1 e^{r_1 t} + c_2 \cdot t \cdot e^{r_1 t}$$

Case 3: We'll discuss this case later

Examples (I think that it is illegal for me to solve the homework problems here, but I'll do some similar problems).

① Problem #6, pag 213

$$y'' - y' - 2y = 0.$$

After doing the substitution $y = e^{rt}$ we end up with the equation

$$r^2 - r - 2 = 0$$

Solving this equation we get $r_1 = 2$, $r_2 = -1$.

$\Rightarrow$ the general solution will be

$$y(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{-t}$$

② Problem #17, pag 213

$$y'' + 2y' + y = 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1$$

Substituting $y = e^{rt}$ we get

$$r^2 + 2r + 1 = 0$$

which has a unique root, $r_1 = -1$. Hence, the general solution is

$$y(t) = C_1 \cdot e^{-t} + C_2 t \cdot e^{-t}$$

Using the Initial Conditions:

$$y(0) = 0 \implies 0 = C_1 e^{-0} + C_2 \cdot 0 \cdot e^{-0} \implies \boxed{0 = C_1}$$

Notice that

$$y'(t) = \left(C_1 e^{-t} + C_2 t e^{-t} \right)'$$

$$= -C_1 \cdot e^{-t} + C_2 \left(e^{-t} - t \cdot e^{-t} \right)$$

$$= e^{-t} \left(C_2 - C_1 - t C_2 \right)$$

$$\therefore \quad y'(t) = e^{-t} \left(C_2 - C_1 - t C_2 \right)$$

And since $y'(0) = 1$

$$\implies \quad 1 = e^{-0} \left(C_2 - C_1 - 0 C_2 \right) \implies 1 = C_2 - C_1$$

But $C_1 = 0 \implies \boxed{C_2 = 1}$

Here

$$y(t) = t \cdot e^{-t}$$

Hint for problem 35: (I won't give out the solution, but I'll try to help).

For the equation $a t^2 y'' + b t y' + c y = 0$ (*), $t > 0$

if we put $y = t^r$

we have that $y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$, and plugging this back in (*) you'll get the desired equation

$$ar(r-1) + br + c = 0$$

NOTE: YOU ARE NOT DONE YET. You still need to show that if $r_1 \neq r_2$, then t^{r_1} and t^{r_2} are linearly independent. (If I were you, I would use Wronskian) Now you can conclude that the general solution of (*) is

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}$$

Section 4.3

Remember that in the last section we didn't consider one of the cases when solving $ay'' + by' + cy = 0$? We are going to consider it now:

First of all, remember that when solving

$$ay'' + by' + cy = 0 \quad (1)$$

we made the substitution $y(t) = e^{rt}$ and obtained the equation

$$ar^2 + br + c = 0 \quad (2)$$

Let's consider now the case when the roots of (2) are nonreal and distinct:

Since the roots of (2) are conjugate, then we can express them as follows:

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, we find that $e^{\alpha t}\sin\beta t$, $e^{\alpha t}\cos\beta t$ are both solutions of (1)

Since they are linearly independent (verify this, math majors) then the general solution of is

$$y(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

Examples

① Problem # 14, Pag 225

$$y'' - y' + y = 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1$$

Substituting $y = e^{rt}$ we end up with the equation

$$r^2 - r + 1 = 0$$

The solutions of this equation are

$$r_1 = \frac{1+\sqrt{3}\,i}{2}$$

$$r_2 = \frac{1-\sqrt{3}\,i}{2}$$

Hence, the general solution is

$$y(t) = e^{\frac{1}{2}t}\left(C_1\cos\left(\frac{\sqrt{3}}{2}t\right) + C_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Notice that (applying the product rule)

$$y'(t) = \frac{1}{2}e^{\frac{1}{2}t}\left(C_1\cos\left(\frac{\sqrt{3}}{2}t\right) + C_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) + e^{\frac{1}{2}t}\left(-\frac{\sqrt{3}\,C_1}{2}\sin\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{\sqrt{3}\,C_2}{2}\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Now, using our initial conditions :

$y(0)=0$

$$\Rightarrow \quad 0 = e^0 \left(c_1 \cos(0) + c_2 \sin(0) \right) \quad \Rightarrow \quad \boxed{0 = c_1}$$

$y'(0)=1$

$$\Rightarrow \quad 1 = \frac{1}{2} e^0 \left(c_1 \cos(0) + c_2 \sin(0) \right) + e^0 \left(-\frac{\sqrt{3}\, c_1}{2} \sin(0) + \frac{\sqrt{3}}{2} c_2 \cos(0) \right)$$

$$\Rightarrow \quad 1 = \frac{c_1}{2} + \frac{\sqrt{3}}{2} c_2$$

Since $c_1 = 0 \Rightarrow \frac{\sqrt{3}}{2} c_2 = 1 \Rightarrow \boxed{c_2 = \frac{2}{\sqrt{3}}}$

Hence the solution is

$$y(t) = \frac{2e^{\frac{1}{2}t}}{\sqrt{3}} \sin\left(\frac{\sqrt{3}\, t}{2} \right)$$

Higher - Order Differential Equations

We are interested in solving homogeneous linear differential equations with constant coefficients. Given an equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0 \quad , \quad (2)$$

if we substitute $y(t) = e^{rt}$ we have the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Here, if a root r is repeated K times, then $e^{rt}, t\cdot e^{rt}, \ldots, t^{K-1}e^{rt}$ are solutions of **(2)** Let's clarify this with an example:

Example:

Find the general solution of

$$\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 0$$

After substituting $y = e^{rt}$ we end up with The equation

$$r^4 + r^3 - 3r^2 - r + 2 = 0$$

$$\Rightarrow (r+1)(r+2)(r-1)^2 = 0$$

So, for the roots $r=-1$ and $r=-2$ we have the solutions:

e^{-t} and e^{-2t}

Since $r=1$ is repeated, then e^t and $t\cdot e^t$ are solutions

Hence, the general solution is

$$y(t) = C_1\cdot e^{-t} + C_2\cdot e^{-2t} + C_3\cdot e^{t} + C_4\, t\cdot e^{t}$$

Section 4.4 : Undetermined Coefficients

Remember when we guessed the particular solutions in 3C? The method of the undetermined coefficients is almost the same, it's really "educated guessing". We are interested in solving equations of the form

$$(3) \quad ay'' + by' + cy = f(t), \quad \text{where } f(t) \neq 0$$

Recall that the operator $L(y) = ay'' + by' + cy$ is <u>linear</u>, so the solution of (3) is of the form $y = y_h + y_p$, where y_h is the homogeneous solution and y_p a particular solution of (3). We already know how to solve the homogeneous equation $ay'' + by' + cy = 0$, so now we are interested in finding the particular solution y_p. That's where guessing plays a big role:

To be honest, there is not much guessing involved here. If $f(t)$ is a polynomial, we should try y_p a polynomial. If $f(t)$ is trigonometric, we should try y_p to be a trigonometric function. If $f(t)$ is exponential, then we should try y_p to be an exponential function. See? Not guessing at all.

Let's see some examples and then discuss certain problems that can arise:

Examples

① Find y_p in:

$$y'' - y' - 2y = 4t^2 \quad (*)$$

Since $f(t) = 4t^2$ is a second-degree polynomial, it makes sense to try $y_p = at^2 + bt + c$. Hence

$$y_p' = 2at + b, \quad y_p'' = 2a$$

Substituting y_p, y_p', y_p'' in $(*)$ we get

$$(2a) - (2at + b) - 2(at^2 + bt + c) = 4t^2$$

$$\Rightarrow \quad t^2(-2a) + t(-2a - 2b) + (2a - b - 2c) = 4t^2$$

From here:

$$\begin{cases} -2a = 4 \\ -2a - 2b = 0 \\ 2a - b - 2c = 0 \end{cases} \Longrightarrow \boxed{\begin{array}{c} a = -2 \\ \hline b = 2 \\ \hline c = -3 \end{array}}$$

Hence, $\boxed{y_p = -2t^2 + 2t - 3}$

② Find y_p in

$$y'' - y' - 2y = t^2 e^t \quad (**)$$

Since $f(t) = t^2 e^t$, then it makes sense to try y_p a polynomial times an exponential, i.e,

$$y_p = e^t (a_2 t^2 + a_1 t + a_0)$$

Note that (Product rule)

$$y_p' = e^t (a_2 t^2 + a_1 t + a_0) + e^t (2a_2 t + a_1)$$

$$\Rightarrow \quad y_p' = e^t \left(a_2 t^2 + (a_1 + 2a_2) t + (a_0 + a_1)\right)$$

Doing this again we obtain

$$y_p'' = e^t \left(a_2 t^2 + (a_1 + 4a_2) t + (a_0 + 2a_1 + 2a_2)\right)$$

After substituting y_p, y_p', y_p'' in $(**)$ we get:

$$e^t \left(a_2 t^2 + (a_1 + 4a_2) t + (a_0 + 2a_1 + 2a_2) - a_2 t^2 - (a_1 + 2a_2) t - (a_0 + a_1) - 2a_2 t^2 - 2a_1 t - 2a_0\right) = t^2 e^t$$

From here, we get

$$\begin{cases} -2a_2 = 1 \\ 2a_2 - 2a_1 = 0 \\ 2a_2 + a_1 - 2a_0 = 0 \end{cases} \Rightarrow \quad \begin{aligned} a_2 &= -1/2 \\ a_1 &= -1/2 \\ a_0 &= -3/4 \end{aligned}$$

$$\Rightarrow \quad y_p = -e^t \left(\frac{t^2}{2} + \frac{t}{2} + \frac{3}{4}\right)$$

Warning: There is a problem when we use this method. Suppose that we want to solve the equation

$$a y'' + b y' + c y = e^{Kt},$$

and $y_h = e^{Kt}$. In this case, you should try $y_p = t \cdot e^{Kt}$ as the particular solution. If it doesn't work, try $y_p = t^2 e^{Kt}$. If it still doesn't work, you are making a mistake so check again.