On the pointwise maximum of convex functions

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Introduction

Let E be a real nontrivial Banach space. If $f : E \to \mathbb{R} \cup \{\infty\}$, we write

$$\operatorname{dom} f := \{x \in E \colon f(x) \in \mathbb{R}\},$$

the "effective domain" of f. We write $\mathcal{PCLSC}(E)$ for the set of all convex lower semicontinuous functions $f\colon E \to \mathbb{R} \cup \{\infty\}$ such that $\operatorname{dom} f \neq \emptyset$. (The "$\mathcal{P}$" stands for "proper", which is the adjective frequently used to denote the fact that the effective domain of a function is nonempty.)

We write E^* for the dual space of E. If $f \in \mathcal{PCLSC}(E)$, we define $f^*\colon E^* \to \mathbb{R} \cup \{\infty\}$ by

$$f^*(x^*) := \sup_E (x^* - f),$$

the *conjugate* of f. Then (see [5], p. 210) $f^* \in \mathcal{PCLSC}(E^*)$.

We define the *biconjugate*, f^{**}, of f by

$$f^{**}(x^{**}) := (f^*)^*(x^{**}) \quad (x^{**} \in E^{**}).$$

From what we have observed above, $f^{**} \in \mathcal{PCLSC}(E^{**})$. In fact, f^{**} is lower semicontinuous with respect to the weak* topology of E^{**} and (see [5], p. 210 again)

$$\text{for all } x \in E, \quad f^{**}(\widehat{x}) = f(x), \tag{0.1}$$

where $\widehat{x}$ is the canonical image of x in E^{**}.

Let $f,\ g \in \mathcal{PCLSC}(E)$. We say that f and g satisfy the *Attouch–Brézis constraint qualification* if

$$\bigcup_{\lambda > 0} \lambda(\operatorname{dom} f - \operatorname{dom} g) \quad \text{is a closed subspace of } E. \tag{$\mathcal{AB}$}$$

It is well known that if $f,\ g \in \mathcal{PCLSC}(E)$ and f and g satisfy ($\mathcal{AB}$) then

$$(f+g)^{**} = f^{**} + g^{**} \text{ on } E^{**}. \tag{0.2}$$

In fact, Rockafellar used the equality (0.2) (under a stronger constraint qualification) in his proof in [5], Proposition 1, p. 211–212 that the subdifferential of an element of $\mathcal{PCLSC}(E)$ is maximal monotone. The equality (0.2) follows easily from the "inf–convolution" formula for $(f+g)^*$, namely that, for all $w^* \in E^*$,

$$(f+g)^*(w^*) = \min_{y^*,\ z^* \in E^*,\ y^*+z^*=w^*} \big[f^*(y^*) + g^*(z^*)\big], \tag{0.3}$$

which was established by Attouch–Brézis in [1], Corollary 2.3, p. 131–132.

In this paper, we consider the corresponding problem with $f+g$ replaced by $f\vee g$, where, for all $f,\ g\in\mathcal{PCLSC}(E)$, $f\vee g$ is defined by

$$(f\vee g)(x):=\max\{f(x),g(x)\}\quad (x\in E).$$

Indeed, we will prove in Theorem 6 that if $f,\ g\in\mathcal{PCLSC}(E)$ and f and g satisfy $(\mathcal{AB})$ then

$$(f\vee g)^{**}=f^{**}\vee g^{**}\ \text{on}\ E^{**}. \tag{0.4}$$

We will complement this in Remark 8 by giving an example showing that the equality (0.4) can fail when $(\mathcal{AB})$ is not satisfied, even if $f\vee g\in\mathcal{PCLSC}(E)$. Now (0.4) would follow easily from the equality that, for all $w^*\in E^*$,

$$(f\vee g)^*(w^*)=\inf_{\rho\in[0,1],\ u^*,\ v^*\in E^*,\ \rho u^*+(1-\rho)v^*=w^*}\big[\rho f^*(u^*)+(1-\rho)g^*(v^*)\big]. \tag{0.5}$$

Unfortunately, (0.5) fails even if $E=\mathbb{R}^2$, $g\in\mathcal{CC}(E)$ and f and g satisfy $(\mathcal{AB})$, where $\mathcal{CC}(E)$ stands for the set of all real convex continuous functions on E. We give an example of this in Remark 3. The actual formula for $(f\vee g)^*$ is much more complicated. In fact, we give two such formulae. The first, in (2.3), appears in Traoré and Volle, [7], Section 7, p. 149 and does not seem to lead easily to (0.4). We now give the background for the second, much more complicated formula, which appears in (2.1), and *does* lead easily to (0.4). Let F be a nontrivial Banach space. (The reason why we also introduce the symbol F to represent a Banach space is that we will be applying these concepts with $F:=E^*$.) If $w\in F$ and $\delta>0$, let $B(w,\delta):=\{x\in F\colon\ \|x-w\|<\delta\}$ and

$$L(w,\delta):=\{(\rho,\sigma,u,v)\colon\ \rho>0,\ \sigma>0,\ u,\ v\in F,\ \rho+\sigma=1,\ \rho u+\sigma v\in B(w,\delta)\}.$$

Suppose that $f,\ g\in\mathcal{PCLSC}(F)$. If $w\in F$, write

$$(f\underset{\delta}{\wedge}g)(w):=\inf_{(\rho,\sigma,u,v)\in L(w,\delta)}\big[\rho f(u)+\sigma g(v)\big]\quad(\delta>0)$$

and

$$(f\underset{0}{\wedge}g)(w):=\sup_{\delta>0}(f\underset{\delta}{\wedge}g)(w)=\lim_{\delta\to0}(f\underset{\delta}{\wedge}g)(w).$$

Then the formula that we shall give in (2.1) is that if $w^*\in E^*$ then

$$(f\vee g)^*(w^*)=(f^*\underset{0}{\wedge}g^*)(w^*).$$

Incidentally, the equality (0.4) is closely related to the result proved by Gossez in [3], Lemme 2.1, p. 375 that the subdifferential of an element of $\mathcal{PCLSC}(E)$ is maximal monotone of "dense type". Unfortunately, it would take us much too far afield to dwell on this issue any further.

Up to this point, we have presented the quantity $(f\underset{0}{\wedge}g)(w)$ simply as a number which appears as the result of certain computations. In fact, $f\underset{0}{\wedge}g$ has much more significance when we consider it as a function. We shall show in Theorem 11 that if $f,\ g \in \mathcal{PCLSC}(F)$ and $\operatorname{dom} f^* \cap \operatorname{dom} g^* \neq \emptyset$ then

$$f \mathop{\wedge}_0 g \in \mathcal{PCLSC}(F) \quad \text{and} \quad (f \mathop{\wedge}_0 g)^* = f^* \vee g^* \text{ on } F^*.$$

In other words, $f \mathop{\wedge}_0 g$ is the "preconjugate" of $f^* \vee g^*$. We shall use this result to give in Theorem 12 a precise description of when (0.4) occurs. Namely, if $f,\ g \in \mathcal{PCLSC}(E)$ and $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ then (0.4) occurs if, and only if,

$$(f \vee g)^* = f^* \mathop{\wedge}_0 g^* \text{ on } E^*.$$

In the proof of Theorem 2, we shall use the minimax theorem below, which follows from a result of Fan (see [2]). (See also [4] and [6] for simple generalizations of Fan's result.)

Theorem 1. *Let A be a nonempty convex subset of a vector space, and B be a nonempty compact convex subset of a topological vector space. Let $h : A \times B \to \mathbb{R}$ be convex on A, and concave and upper semicontinuous on B. Then*

$$\inf_A \max_B h = \max_B \inf_A h.$$

The conjugate of a maximum

Theorem 2. *Suppose that $f,\ g \in \mathcal{PCLSC}(E)$, f and g satisfy $(\mathcal{AB})$ and $w^* \in E^*$. Then:*

$$(f \vee g)^*(w^*) = (f^* \mathop{\wedge}_0 g^*)(w^*). \tag{2.1}$$

Proof. We first prove that if $\rho,\ \sigma > 0$ then there exist $u^*,\ v^* \in E^*$ such that

$$\rho u^* + \sigma v^* = w^* \quad \text{and} \quad \rho f^*(u^*) + \sigma g^*(v^*) = \sup_A \big[w^* - \rho f - \sigma g\big], \tag{2.2}$$

where A is the nonempty convex set $\operatorname{dom} f \cap \operatorname{dom} g$. To this end, let $\rho,\ \sigma > 0$. Clearly ρf and σg also satisfy $(\mathcal{AB})$ consequently, from the Attouch–Brézis formula for the conjugate of a sum (see (0.3) above), there exist $y^* \in E^*$ and $z^* \in E^*$ such that

$$y^* + z^* = w^* \quad \text{and} \quad (\rho f)^*(y^*) + (\sigma g)^*(z^*) = (\rho f + \sigma g)^*(w^*).$$

We now put $u^* := y^*/\rho$ and $v^* := z^*/\sigma$, and obtain (2.2) since we then have $(\rho f)^*(y^*) = \rho f^*(u^*)$, $(\sigma g)^*(z^*) = \sigma g^*(v^*)$ and $(\rho f + \sigma g)^*(w^*) = \sup_A \big[w^* - \rho f - \sigma g\big]$.

We next prove that

$$(f \vee g)^*(w^*) = \min_{\lambda \in [0,1]} \sup_A \big[w^* - \lambda f - (1-\lambda) g\big]. \tag{2.3}$$

This follows from the minimax theorem, Theorem 1, with $B := [0,1]$, since

$$\begin{aligned}(f \vee g)^*(w^*) &= \sup_{x \in A} \big[\langle x, w^* \rangle - (f \vee g)(x)\big] \\ &= \sup_{x \in A} \min_{\lambda \in [0,1]} \big[\langle x, w^* \rangle - \lambda f(x) - (1-\lambda) g(x)\big].\end{aligned}$$

We now prove the inequality "$\geq$" in (2.1). Since this trivially true if $(f \vee g)^*(w^*) = \infty$, we can and will suppose that $(f \vee g)^*(w^*) \in \mathbb{R}$. Let $\delta,\ \varepsilon > 0$. We shall prove that there exists $(\rho, \sigma, u^*, v^*) \in L(w^*, \delta)$ such that

$$\rho f^*(u^*) + \sigma g^*(v^*) \leq (f \vee g)^*(w^*) + \varepsilon. \tag{2.4}$$

The desired inequality will then follow by taking the infimum over $(\rho, \sigma, u^*, v^*) \in L(w^*, \delta)$ and then letting $\delta \to 0$ and $\varepsilon \to 0$. From (2.3), there exists $\lambda \in [0,1]$ such that

$$\sup_A \big[w^* - \lambda f - (1-\lambda) g\big] = (f \vee g)^*(w^*).$$

Case 1. $(\lambda \in (0,1))$ From (2.2), there exist $u^*,\ v^* \in E^*$ such that

$$\lambda u^* + (1-\lambda) v^* = w^* \quad \text{and} \quad \lambda f^*(u^*) + (1-\lambda) g^*(v^*) = (f \vee g)^*(w^*)$$

and (2.4) is immediate with $\rho := \lambda$ and $\sigma := 1 - \lambda$.
Case 2. $(\lambda = 0)$ Here we have

$$\sup_A \big[w^* - g\big] = (f \vee g)^*(w^*). \tag{2.5}$$

As we have already observed, $f^* \in \mathcal{PCLSC}(E^*)$. Hence there exists $x^* \in E^*$ such that $f^*(x^*) \in \mathbb{R}$. If $\rho > 0$, $\sigma > 0$, $\rho + \sigma = 1$ and (ρ, σ) is sufficiently close to $(0,1)$ then

$$(\rho, \sigma, x^*, w^*) \in L(w^*, \delta) \quad \text{and} \quad \rho f^*(x^*) \leq \rho (f \vee g)^*(w^*) + \varepsilon. \tag{2.6}$$

Using (2.2) again, there exist $u^*,\ v^* \in E^*$ such that

$$\rho u^* + \sigma v^* = \rho x^* + \sigma w^* \tag{2.7}$$

and

$$\begin{aligned}\rho f^*(u^*) + \sigma g^*(v^*) &= \sup_A \big[\rho x^* + \sigma w^* - \rho f - \sigma g\big] \\ &= \sup_A \big[\rho(x^* - f) + \sigma(w^* - g)\big] \\ &\leq \rho \sup_A \big[x^* - f\big] + \sigma \sup_A \big[w^* - g\big] \\ &\leq \rho f^*(x^*) + \sigma \sup_A \big[w^* - g\big].\end{aligned}$$

Thus, from (2.6) and (2.5),

$$\begin{aligned}\rho f^*(u^*) + \sigma g^*(v^*) &\leq \big[\rho (f \vee g)^*(w^*) + \varepsilon\big] + \sigma (f \vee g)^*(w^*) \\ &= (f \vee g)^*(w^*) + \varepsilon.\end{aligned}$$

We now obtain (2.4) since, from (2.6) and (2.7), $(\rho, \sigma, u^*, v^*) \in L(w^*, \delta)$.

Case 3. ($\lambda = 1$) The proof of this is similar to that of Case 2, except that the roles of f and g are reversed. This completes the proof of the inequality "$\ge$" in (2.1).

We now prove the reverse inequality. Let $x \in A$ and $(\rho, \sigma, u^*, v^*) \in L(w^*, \delta)$. Then

$$\begin{aligned}\rho f^*(u^*) + \sigma g^*(v^*) &\ge \rho\big[\langle x, u^*\rangle - f(x)\big] + \sigma\big[\langle x, v^*\rangle - g(x)\big]\\ &= \langle x, \rho u^* + \sigma v^*\rangle - \rho f(x) - \sigma g(x)\\ &\ge \langle x, w^*\rangle - \delta\|x\| - (f \vee g)(x).\end{aligned}$$

Taking the infimum over $(\rho, \sigma, u^*, v^*) \in L(w^*, \delta)$, we obtain

$$(f^* \underset{\delta}{\wedge} g^*)(w^*) \ge \langle x, w^*\rangle - \delta\|x\| - (f \vee g)(x).$$

Letting $\delta \to 0$,

$$(f^* \underset{0}{\wedge} g^*)(w^*) \ge \langle x, w^*\rangle - (f \vee g)(x).$$

The inequality "$\le$" in (2.1) now follows by taking the supremum of the right hand side over $x \in A$. (Note: this can also be deduced from Lemma 10(a), which is independent of the analysis in this Theorem.)

This completes the proof of Theorem 2. ▌

If $C \subset E$, the *indicator function* of C is the function $I_C: E \to \mathbb{R} \cup \{\infty\}$ defined by

$$I_C(x) := \begin{cases} 0 & \text{if } x \in C;\\ \infty & \text{otherwise.}\end{cases}$$

Remark 3. We now give the promised example where $f,\ g \in \mathcal{PCLSC}(E)$ and f and g satisfy $(\mathcal{AB})$, but (0.5) fails. (We leave it to the reader to check that (0.5) *does* hold if both $f \in \mathcal{CC}(E)$ and $g \in \mathcal{CC}(E)$.) Here is the example. Define $f \in \mathcal{PCLSC}(\mathbb{R}^2)$ and $g \in \mathcal{CC}(\mathbb{R}^2)$ by

$$f(x_1, x_2) := \begin{cases} x_2 & \text{if } x_1 \ge 0;\\ \infty & \text{otherwise;}\end{cases}$$

and

$$g(x_1, x_2) := x_1.$$

Then $(f \vee g)^*(0) = -\inf(f \vee g) = 0$. On the other hand, f^* is the indicator function of $(-\infty, 0] \times \{1\}$ and g^* is the indicator function of $\{(1, 0)\}$. Consequently, if $\rho \in [0, 1]$, $u^* \in \mathbb{R}^2$, $v^* \in \mathbb{R}^2$ and $\rho u^* + (1-\rho)v^* = 0$ then $\rho f^*(u^*) + (1-\rho)g^*(v^*) = \infty$, and so (0.5) fails. We note that $(\mathcal{AB})$ is satisfied in this example because $g \in \mathcal{CC}(\mathbb{R}^2)$.

Remark 4. Let $f,\ g \in \mathcal{PCLSC}(E)$, $f,\ g$ satisfy $(\mathcal{AB})$, $x \in E$ and $f(x) = g(x) \in \mathbb{R}$. We discuss briefly the problem of finding a formula for $\partial(f \vee g)(x)$. Suppose first that, for all $w^* \in \partial(f \vee g)(x)$, the following "exact" version of (0.5) holds:

$$(f \vee g)^*(w^*) = \min_{\rho \in [0,1],\ u^*,\ v^* \in E^*,\ \rho u^* + (1-\rho)v^* = w^*} \big[\rho f^*(u^*) + (1-\rho)g^*(v^*)\big]. \qquad (4.1)$$

Then it is easily seen that

$$\partial(f \vee g)(x) = \operatorname{co}(\partial f(x) \cup \partial g(x)).$$

In general, we have the formulae for $(f \vee g)^*(w^*)$ given by (2.1) and (2.3), and we have the formula established by Volle in [8], Théorème 2, p. 848 that

$$\partial(f \vee g)(x) = \operatorname{co}(\partial f(x) \cup \partial g(x)) + N_{\operatorname{dom} f}(x) + N_{\operatorname{dom} g}(x), \tag{4.2}$$

where "$N_C(x)$" stands for the normal cone to C at x. However, we do not know an easy way of deducing (4.2) from (2.1) or (2.3).

The biconjugate of a maximum

It is an easy consequence of the definitions that if $f \in \mathcal{PCLSC}(E)$ then

$$t^{**} \in E^{**},\ f^{**}(t^{**}) \le 0 \text{ and } w^* \in E^* \quad \Longrightarrow \quad \langle w^*, t^{**}\rangle \le f^*(w^*). \tag{4.3}$$

Lemma 5. *Suppose that $f,\ g \in \mathcal{PCLSC}(E)$, f and g satisfy ($\mathcal{AB}$) and also that $f^{**}(t^{**}) \vee g^{**}(t^{**}) \le 0$.*
(a) Let $w^* \in E^*$. Then $\langle w^*, t^{**}\rangle \le (f \vee g)^*(w^*)$.
(b) $(f \vee g)^{**}(t^{**}) \le 0$.

Proof. (a) Let $\delta > 0$. If $(\rho, \sigma, u^*, v^*) \in L(w^*, \delta)$ then, using (4.3),

$$\begin{aligned} \rho f^*(u^*) + \sigma g^*(v^*) &\ge \rho\langle u^*, t^{**}\rangle + \sigma\langle v^*, t^{**}\rangle \\ &= \langle \rho u^* + \sigma v^*, t^{**}\rangle \\ &\ge \langle w^*, t^{**}\rangle - \delta\|t^{**}\|. \end{aligned}$$

Thus, taking the infimum over $(\rho, \sigma, u^*, v^*) \in L(w^*, \delta)$,

$$(f^* \mathop{\wedge}_{\delta} g^*)(w^*) \ge \langle w^*, t^{**}\rangle - \delta\|t^{**}\|,$$

and (a) now follows from Theorem 2 by letting $\delta \to 0$. (b) is immediate from (a). ▌

Theorem 6. *Suppose that $f,\ g \in \mathcal{PCLSC}(E)$, and f and g satisfy ($\mathcal{AB}$). Then*

$$(f \vee g)^{**} = f^{**} \vee g^{**} \text{ on } E^{**}.$$

Proof. We first prove that if $t^{**} \in E^{**}$ then

$$(f \vee g)^{**}(t^{**}) \le f^{**}(t^{**}) \vee g^{**}(t^{**}). \tag{6.1}$$

Let $\alpha := f^{**}(t^{**}) \vee g^{**}(t^{**})$. Since (6.1) is immediate if $\alpha = \infty$, we can and will suppose that $\alpha \in \mathbb{R}$. Then (6.1) follows from Lemma 5(b) with f replaced by $f - \alpha$ and g replaced by $g - \alpha$.

Since $f \vee g \ge f$ on E, $(f \vee g)^{**} \ge f^{**}$ on E^{**}. Similarly, $(f \vee g)^{**} \ge g^{**}$ on E^{**}, and so $(f \vee g)^{**} \ge f^{**} \vee g^{**}$ on E^{**}. The result now follows from (6.1). ▌

Corollary 7. *Let $g_0 \in \mathcal{PCLSC}(E)$ and $g_1, \ldots, g_m \in \mathcal{CC}(E)$. Then*

$$(g_0 \vee \cdots \vee g_m)^{**} = {g_0}^{**} \vee \cdots \vee {g_m}^{**}.$$

Proof. This is immediate from Theorem 6 and induction. ▌

Remark 8. We now give an example showing that (0.4) can fail when $(\mathcal{AB})$ is not satisfied, even if $f \vee g \in \mathcal{PCLSC}(E)$. (The conclusion of Theorem 2 must also fail for this example, as we shall see in Theorem 12.) Let $E = c_0$,

$$C := \big\{\{x_n\}_{n\ge1} \in c_0\colon\ x_1 \ge x_2 \ge x_3 \ge \ldots 0\big\},$$
$$D := \Big\{\{x_n\}_{n\ge1} \in c_0\colon\ \sum_{n=1}^{\infty} \frac{1}{2^n}(x_1 - x_{n+1}) = 0\Big\},$$

and define $f,\ g \in \mathcal{PCLSC}(E)$ by $f := I_C$ and $g := I_D$. Now if $x \in C \cap D$ then

$$\sum_{n=1}^{\infty} \frac{1}{2^n}(x_1 - x_{n+1}) = 0 \quad \text{and,} \quad \text{for all } n \ge 1, \quad x_1 - x_{n+1} \ge 0.$$

It follows that, for all $n \ge 1$, $x_1 - x_{n+1} = 0$, and so x is a constant sequence. Since $x \in c_0$, we deduce that $x = 0$. These observations lead easily to the conclusion that $f \vee g = I_{\{0\}}$, from which $(f \vee g)^* = 0$ and $(f \vee g)^{**} = I_{\{0\}}$ (relative to E^{**}). In particular, if $e := (1, 1, 1, \ldots) \in \ell^\infty = E^{**}$, then

$$(f \vee g)^{**}(e) = \infty. \tag{8.1}$$

If $m \ge 1$, define y^m and $z^m \in E$ as follows:

$${y^m}_n := \begin{cases} 1 & \text{if } n \le m; \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad {z^m}_n := \begin{cases} 1 & \text{if } n \le m; \\ 2 & \text{if } n = m+1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $y^m \in C$ and $z^m \in D$, from which $f(y^m) = 0$ and $g(z^m) = 0$. Using (0.1), we deduce from this that $f^{**}(\widehat{y^m}) = 0$ and $g^{**}(\widehat{z^m}) = 0$. Since $\widehat{y^m} \to e$ and $\widehat{z^m} \to e$ in the weak* topology of E^{**} as $m \to \infty$, and f^{**} and g^{**} are weak* lower semicontinuous, it follows that $f^{**}(e) \le 0$ and $g^{**}(e) \le 0$, from which

$$(f^{**} \vee g^{**})(e) \le 0. \tag{8.2}$$

If we now combine (8.1) and (8.2), we see that (0.4) fails, as claimed.

The preconjugate of a maximum

Lemma 9. *Suppose that $f,\ g \in \mathcal{PCLSC}(F)$ with $\operatorname{dom} f^* \cap \operatorname{dom} g^* \neq \emptyset$, and $\delta > 0$.*
(a) Let $x^* \in F^*$. Then $f \underset{\delta}{\wedge} g \geq x^* - (f^* \vee g^*)(x^*) - \delta\|x^*\|$ on F.
(b) $f \underset{\delta}{\wedge} g\colon F \to \mathbb{R} \cup \{\infty\}$.
(c) $f \underset{\delta}{\wedge} g \leq f$ on F and $f \underset{\delta}{\wedge} g \leq g$ on F.
(d) $f \underset{\delta}{\wedge} g$ is convex.

Proof. (a) Since the result is trivial if $(f^* \vee g^*)(x^*) = \infty$, we can and will suppose that $(f^* \vee g^*)(x^*) \in \mathbb{R}$. Let $w \in E$ and (ρ, σ, u, v) be an arbitrary element of $L(w, \delta)$. Then

$$\begin{aligned}\rho f(u) + \sigma g(v) &\geq \rho\big[\langle u, x^*\rangle - f^*(x^*)\big] + \sigma\big[\langle v, x^*\rangle - g^*(x^*)\big]\\ &\geq \langle \rho u + \sigma v, x^*\rangle - f^*(x^*) \vee g^*(x^*)\\ &\geq \langle w, x^*\rangle - \delta\|x^*\| - (f^* \vee g^*)(x^*).\end{aligned}$$

We now obtain (a) by taking the infimum over $(\rho, \sigma, u, v) \in L(w, \delta)$.
(b) This follows from (a) by taking $x^* \in \operatorname{dom} f^* \cap \operatorname{dom} g^*$.
(c) We shall prove that $f \underset{\delta}{\wedge} g \leq f$ on F, the proof that $f \underset{\delta}{\wedge} g \leq g$ on F is similar. So let $w \in F$. We need to show that

$$(f \underset{\delta}{\wedge} g)(w) \leq f(w). \tag{9.1}$$

Since this is trivial if $f(w) = \infty$, we can and will suppose that $w \in \operatorname{dom} f$. Fix $v \in \operatorname{dom} g$. If $\rho > 0$, $\sigma > 0$, $\rho + \sigma = 1$ and (ρ, σ) is sufficiently close to $(1, 0)$ then $(\rho, \sigma, w, v) \in L(w, \delta)$ and so $\rho f(w) + \sigma g(v) \geq (f \underset{\delta}{\wedge} g)(w)$. We now obtain (9.1) by letting $(\rho, \sigma) \to (1, 0)$.

(d) For $i = 1,\ 2$, let $w_i \in E$, $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. Put $w_3 := \sum_i \lambda_i w_i$. We shall prove that

$$\textstyle\sum_i \lambda_i (f \underset{\delta}{\wedge} g)(w_i) \geq (f \underset{\delta}{\wedge} g)(w_i), \tag{9.2}$$

which will give the reqired result. To this end, let $(\rho_i, \sigma_i, u_i, v_i)$ be arbitrary elements of $L(w_i, \delta)$. It is easy to check that

$$\textstyle\sum \lambda_i(\rho_i u_i + \sigma_i v_i) \in B(w_3, \delta). \tag{9.3}$$

Put $\rho_3 := \sum_i \lambda_i \rho_i \in (0, 1)$, $\sigma_3 := \sum_i \lambda_i \sigma_i \in (0, 1)$, $u_3 := \sum_i \lambda_i \rho_i u_i / \rho_3 \in F$ and $v_3 := \sum_i \lambda_i \sigma_i v_i / \sigma_3 \in F$. Since $\rho_3 + \sigma_3 = 1$, it follows from these definitions that

$$\textstyle\sum_i \lambda_i \rho_i f(u_i) \geq \rho_3 f(u_3) \quad \text{and} \quad \sum_i \lambda_i \sigma_i g(v_i) \geq \sigma_3 g(v_3).$$

Consequently,

$$\textstyle\sum_i \lambda_i \big[\rho_i f(u_i) + \sigma_i g(v_i)\big] \geq \rho_3 f(u_3) + \sigma_3 g(v_3). \tag{9.4}$$

We also derive from (9.3) that $\rho_3 u_3 + \sigma_3 v_3 \in B(w_3, \delta)$. Combining this with (9.4), we obtain

$$\textstyle\sum_i \lambda_i \big[\rho_i f(u_i) + \sigma_i g(v_i)\big] \geq (f \underset{\delta}{\wedge} g)(w_3),$$

and (9.2) now follows by taking the infima over $(\rho_i, \sigma_i, u_i, v_i) \in L(w_i, \delta)$. ▌

Lemma 10. *Suppose that* $f,\ g \in \mathcal{PCLSC}(F)$ *and* $\operatorname{dom} f^* \cap \operatorname{dom} g^* \neq \emptyset$.
(a) *Let* $x^* \in F^*$. *Then* $x^* - f \mathop{\wedge}\limits_{0} g \le (f^* \vee g^*)(x^*)$ *on* F.
(b) $f \mathop{\wedge}\limits_{0} g \le f$ *on* F *and* $f \mathop{\wedge}\limits_{0} g \le g$ *on* F.

Proof. These assertions follow easily from Lemma 9 by letting $\delta \to 0$. ▌

Theorem 11. *Suppose that* $f,\ g \in \mathcal{PCLSC}(F)$ *and* $\operatorname{dom} f^* \cap \operatorname{dom} g^* \neq \emptyset$. *Then*

$$f \mathop{\wedge}\limits_{0} g \in \mathcal{PCLSC}(F) \quad \text{and} \quad (f \mathop{\wedge}\limits_{0} g)^* = f^* \vee g^* \text{ on } F^*.$$

Proof. It is clear from Lemma 9(a) by letting $\delta \to 0$ that $(f \mathop{\wedge}\limits_{0} g)\colon E \to \mathbb{R} \cup \{\infty\}$ and is convex. In order to show that $f \mathop{\wedge}\limits_{0} g \in \mathcal{PCLSC}(F)$, it only remains to prove that $f \mathop{\wedge}\limits_{0} g$ is lower semicontinuous on F. To this end, let $w \in F$ and $\alpha < (f \mathop{\wedge}\limits_{0} g)(w)$. We can choose $\delta > 0$ so that $\alpha < (f \mathop{\wedge}\limits_{\delta} g)(w)$. Let $\eta := \delta/2$. Since

$$x \in B(w, \eta) \quad \Longrightarrow \quad B(x, \eta) \subset B(w, \delta),$$

it follows by taking the appropriate infima that

$$x \in B(w, \eta) \quad \Longrightarrow \quad (f \mathop{\wedge}\limits_{\eta} g)(x) \ge (f \mathop{\wedge}\limits_{\delta} g)(w).$$

Hence

$$x \in B(w, \eta) \quad \Longrightarrow \quad (f \mathop{\wedge}\limits_{0} g)(x) > \alpha.$$

This gives the required lower semicontinuity. It follows from Lemma 10(b) that $(f \mathop{\wedge}\limits_{0} g)^* \ge f^*$ on F^* and $(f \mathop{\wedge}\limits_{0} g)^* \ge g^*$ on F^*, from which $(f \mathop{\wedge}\limits_{0} g)^* \ge f^* \vee g^*$ on F^*. The opposite inequality follows by taking the supremum over F in Lemma 10(a). ▌

Theorem 12. *Suppose that* $f,\ g \in \mathcal{PCLSC}(E)$ *and* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then*

$$(f \vee g)^{**} = f^{**} \vee g^{**} \text{ on } E^{**} \iff (f \vee g)^* = f^* \mathop{\wedge}\limits_{0} g^* \text{ on } E^*.$$

Proof. We first note that $\operatorname{dom} f^{**} \cap \operatorname{dom} g^{**} \neq \emptyset$ hence, from Lemma 10(c) and Theorem 11 with $F := E^*$ and f and g replaced by f^* and g^*,

$$f^* \mathop{\wedge}\limits_{0} g^* \in \mathcal{PCLSC}(E^*) \quad \text{and} \quad (f^* \mathop{\wedge}\limits_{0} g^*)^* = f^{**} \vee g^{**} \text{ on } E^{**}. \tag{12.1}$$

It is immediate from this that

$$(f \vee g)^* = f^* \mathop{\wedge}\limits_{0} g^* \text{ on } E^* \quad \Longrightarrow \quad (f \vee g)^{**} = f^{**} \vee g^{**} \text{ on } E^{**}.$$

Now suppose that $(f \vee g)^{**} = f^{**} \vee g^{**}$ on E^{**}. From (12.1), $(f \vee g)^{**} = (f^* \underset{0}{\wedge} g^*)^*$ on E^{**}, and consequently

$$(f \vee g)^{***} = (f^* \underset{0}{\wedge} g^*)^{**} \text{ on } E^{***}.$$

Since both $(f \vee g)^*$ and $(f^* \underset{0}{\wedge} g^*)$ are in $\mathcal{PCLSC}(E^*)$, it follows from (0.1) (with E replaced by E^*) that

$$(f \vee g)^* = f^* \underset{0}{\wedge} g^* \text{ on } E^*,$$

as required. ▌

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