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Minimax and Monotonicity

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Introduction

The primary purpose of these notes is to collect together in one place a number of results that have been proved in recent years about monotone multifunctions on a (possibly nonreflexive) Banach space.

Many of these results involve finding an element of a dual space satisfying certain properties. Ultimately, the solution of such problems relies on the Hahn–Banach theorem. However, it is frequently not easy to apply the Hahn–Banach theorem directly, since this involves the manipulation of sublinear functionals defined by extremely cumbersome formulae.

It turns out that the most convenient way of applying the Hahn–Banach theorem for the problems that we will be considering is through the vehicle of a *minimax theorem*. In fact, one can formulate this procedure into what we will call the "minimax technique". The idea behind this is to use the minimax theorem, the one-dimensional Hahn–Banach theorem and the Banach–Alaoglu theorem to transform problems on the existence of elements of the dual space into problems on the existence of a certain real constant. We will use this technique many times. However, we must emphasize that the minimax theorem as we use it is essentially a repackaging of the Hahn–Banach theorem.

So our secondary purpose is to describe the use of the minimax theorem as a functional analytic tool. The minimax technique does have one great virtue: in many cases one can work backwards from a conjecture, find whether it is reasonable and, if it is, obtain a proof. Even when a result is already known, the proof of it obtained from the minimax technique is usually as simple, if not simpler, than that obtained from more "usual" techniques. Furthermore, isofar as a study of monotonicity is concerned, we feel that the additional work associated with an understanding of the minimax technique is more than justified by the additional insight that this approach permits.

The first result on monotone multifunctions that we will consider will be Rockafellar's "surjectivity" characterization of those monotone multifunctions on a reflexive space that are maximal. In order to do this, we will introduce the "big convexification" of a multifunction, a concept that will be extremely useful to us all through these notes.

The analysis of convex lower semicontinuous functions is, by and large, simpler than the analysis of multifunctions. It is with this in mind that we show how to associate convex lower semicontinuous functions with any multifunction. It turns out that these functions capture enough of the structure of the multifunction for us to obtain simple proofs of a number of the known results about monotone multifunctions on (possibly nonreflexive) Banach spaces. For instance, we give simple proofs that *any nontrivial monotone multifunction is locally bounded at any absorbing point of its domain* and also, using the minimax technique, that *the interior of the domain of any maximal monotone multifunction is convex.* In fact, we can give a *precise description* of this interior in terms of the "essential domain" of one of the associated convex functions mentioned above. We can also prove the stronger result that *any point surrounded by the domain of a maximal monotone multifunction is necessarily an interior point of that domain.*

Since we do not assume that the reader has any prior knowledge of convex analysis, we take time off to establish the results that we will need in this area, some of which depend ultimately on Baire's theorem. In particular, we introduce the "dom–dom lemma", a generalization of the classical open mapping theorem.

Using the dom–dom lemma and the minimax technique, we discuss one of the most fascinating questions about monotone multifunctions: *when is the sum of maximal monotone multifunctions on a reflexive Banach space maximal monotone*. We give a proof of Rockafellar's original result, and unify a number of other results that have been proved in this direction recently. In fact, we will give a necessary and sufficient for the sum of maximal monotone multifunctions on a reflexive Banach space to be maximal monotone.

We will discuss some of the subclasses of the class of maximal monotone multifunctions that have been introduced over the years. The oldest of them, the maximal monotone multifunctions of "type (D)", dates back to 1971, while those that are of "type (FP)", "type (FPV)", "type (NI)", "type (ANA)" and those that are "strongly maximal monotone" are much more recent. We give the definitions of these subclasses, prove the results known about them, and also discuss a number of related open problems.

Rockafellar also proved that *the subdifferential of a somewhere finite convex lower semicontinuous function is maximal monotone*. We give a proof of this result that relies on Ekeland's variational principle and the minimax technique, and then consider generalizations of this result related to the subclasses of multifunctions discussed above.

Using the minimax technique many times, we also consider (possibly unbounded) positive linear operators from a Banach space into its dual. We give a criterion for such an operator to be maximal monotone, and discuss the relationship between such operators and the subclasses of multifunctions discussed above.

In Chapter I, we give the results from functional analysis on which these notes will be based. Starting from the Hahn-Banach theorem for sublinear functionals, we deduce in Theorem 1.1(b) the version of it due to Mazur and Orlicz, an extremely useful result that is not nearly as well known as it deserves to be. We prove the minimax theorem in Theorem 3.1. In Section 4, we introduce the results from Banach space theory that we shall need. The most important of these is the Banach–Alaoglu theorem, Theorem 4.1. Thus Sections 1–4 contain everything we need in order to use the minimax technique. We also define reflexivity in Section 4. In general terms, one of the big problems about monotone multifunctions is finding when results that are known in the reflexive case can be extended to the nonreflexive case. This is why we have included Section 5. In this section, we show that there are "natural boundaries" to minimax theorems in the sense that if a nonempty bounded closed convex set in a Banach space has the property that minimax theorems always hold on it (in a reasonable sense) then the set is necessarily weakly compact. If this set is a closed ball then, from Theorem 4.3, the Banach space is necessarily reflexive.

Section 6 is about the minimax technique. The results in this section will not be used until Section 33. We start off Section 6 by reproving two results that most readers will already be familiar with, the extension form of the Hahn–Banach theorem and the "point – closed convex set" separation theorem in a Banach space. The second of these gives our first concrete example of how the minimax technique can be used to transform a problem on the existence of a linear functional into a problem on the existence of a real constant. The next two examples of the minimax technique will not be familiar to readers who do not have a background in convex analysis. In the second of these, Example 6.2, we give a necessary *and sufficient* form of the Fenchel duality theorem. The minimax technique approach avoids the aggravating problem of the "vertical hyperplane" that so destroys the elegance of the usual approach through the Eidelheit separation theorem. Section 6 closes with some additional remarks on Fenchel conjugates of convex functions.

So now we have transformed our problem on the existence of a linear functional into a problem on the existence of a real constant, how do we find the real constant? What leaps to mind is, of course, Baire's theorem. This is frequently the case — in fact, Chapter III will discuss some new results on convex functions that have been spawned by this problem. However, there is a significant case where this constant can be produced without Baire's theorem. This will be dealt with in Section 7. The main result here is the "fg–theorem" Theorem 7.2. The fg–theorem is unusual in that it uses the minimax theorem twice, first to produce a real constant and then, using this constant as a bound, to apply the minimax technique as described above. Though monotonicity is not mentioned in it, the fg–theorem is, in fact, an abstraction of results on monotonicity that appeared in our paper [54]. Thus Section 7 is a bridge between functional analysis and monotonicity.

In Chapter II, we give the definitions and develop the machinery that we will use to prove one direction of Rockafellar's surjectivity theorem (see below). We define multifunctions, monotonicity and maximality formally in Section 8. For some problems, it is convenient to think of a multifunction from E into 2^{E^*} (for the rest of this introduction, E is a nonzero real Banach space) as a subset of $E \times E^*$. This is pursued in Section 9, where we introduce a *big convexification* of any nonempty subset of $E \times E^*$ and the associated linear operators p, q and r. We also prove the "pqr–lemma", in which monotone subsets of $E \times E^*$ are characterized in terms of their big convexifications. We will use the pqr–lemma many times in these notes.

Section 10 is about reflexive spaces. In Lemma 10.1, we apply the minimax technique to obtain an equivalence valid for any nonempty subset of $E \times E^*$. We use this in Theorem 10.6 to prove that *if M is a monotone subset of $E \times E^*$ then*

$$M \text{ is maximal monotone} \iff M + G(-J) = E \times E^*,$$

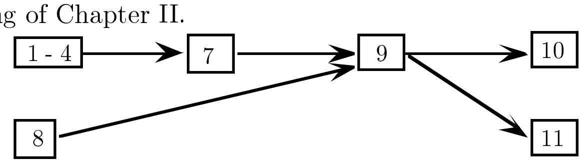
where J is the duality map, and we deduce in Theorem 10.7 that

$$S\colon E \mapsto 2^{E^*} \text{ is maximal monotone} \quad \Longrightarrow \quad R(S + J) = E^*.$$

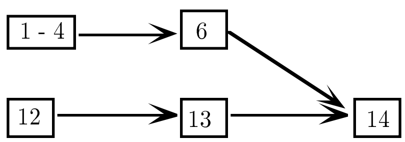
This is one direction of Rockafellar's "surjectivity theorem", except that our results hold for any reflexive space, not only ones where the norms of E and E^* are strictly convex. In the final section of this chapter, Section 11, we use the minimax technique to prove the following useful result:

$$S\colon E \mapsto 2^{E^*} \text{ maximal monotone and } R(S) \text{ bounded} \quad \Longrightarrow \quad D(S) = E.$$

The flowchart below should serve to show which sections are needed for an understanding of Chapter II.



In Chapter III, we leave our discussion of monotonicity temporarily, and turn our attention to convex analysis. We collect together in Sections 12 and 13 various results on convex functions that depend ultimately on Baire's theorem. The "dom lemma", Lemma 12.2, is a generalization to convex functions of the classical uniform boundedness (Banach Steinhaus) theorem (see Remark 12.4) and the "dom–dom lemma", Lemma 13.1 is a generalization to convex functions of the classical open mapping theorem (see Remark 13.3). Both of these results will be applied later on to obtain bounds that can be used for the minimax technique. (We should, however, remind the reader that the bound β of the fg–theorem, Theorem 7.2 was found *without* the use of Baire's theorem.) We can think of the dom lemma and the dom–dom lemma as "quantitative" results, since their main purpose is to provide numerical bounds. Associated with them are two "qualitative" results, the "dom corollary", Corollary 12.3, and the "dom–dom corollary", Corollary 13.2, from which the numerics have been removed. These results will also be of use to us later on. We give in Remark 13.4 a brief discussion of convex Borel sets and functions. In the final section of this chapter, Section 14, we show how the dom–dom lemma leads to the Attouch–Brézis version of the Fenchel duality theorem, which we state formally as Theorem 14.2. Theorem 14.2 will not be used in our later work on monotonicity, however we thought that it would be appropriate to show how the dom–dom lemma enables us to complete Example 6.2. The next flowchart shows which sections are needed for an understanding of Chapter III.



In Chapter IV, we end our digression into convex analysis, and return to our analysis of multifunctions. As we have already explained, we can get considerable insight into the properties of multifunctions by considering associated convex functions (and then applying the results of the Chapter III to them). In Section 15, we define two such convex functions ψ_S and χ_S associated with every nontrivial multifunction $S\colon\ E \mapsto 2^{E^*}$. The formula for χ_S is more complicated than that for ψ_S. However, the set indexing χ_S is convex, which will enable us to use the minimax technique. We continue this analysis in Section 16 by showing how ψ_S and χ_S interact with closed convex sets and closed subspaces. The result is particularly simple when S is maximal monotone and $D(S)$ is closed and convex — we shall show in Theorem 16.2 that, in this case,

$$\operatorname{dom}\chi_S = \operatorname{dom}\psi_S = D(S).$$

(The proof, however, is not so simple: it uses either Rockafellar's result on the maximal monotonicity of subdifferentials, or the Bishop–Phelps theorem.) The remainder of Section 16 is devoted to proving some results that we shall need for our analysis of constraint qualifications for pairs of maximal monotone multifunctions. In particular, in Theorem 16.10, we deal with the problem of "restricting" a maximal monotone multifunction to a closed subspace.

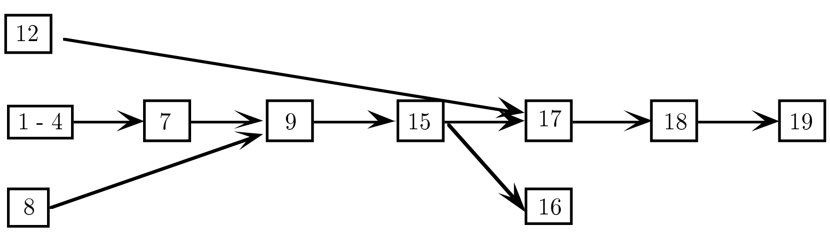
In Section 17, using the dom lemma, we establish results that we will use in Sections 18 and 20, and we also prove a local boundedness theorem for any (not necessarily monotone) multifunction on a Banach space. Specifically, we prove in Theorem 17.3 that *a nontrivial multifunction is locally bounded at each point surrounded by* $\operatorname{dom}\psi_S$. This extends the results known for monotone multifunctions. In Section 18, we use the minimax technique to prove two main results, the "six set theorem", Theorem 18.3, and the "nine set theorem", Theorem 18.4. For both of these, we assume that S is maximal monotone. In the six set theorem, we prove that the six sets $\operatorname{int} D(S)$, $\operatorname{int}(\operatorname{co} D(S))$, $\operatorname{int}(\operatorname{dom} \chi_S)$, $\operatorname{sur} D(S)$, $\operatorname{sur}(\operatorname{co} D(S))$ and $\operatorname{sur}(\operatorname{dom} \chi_S)$ coincide, and in the nine set theorem, we prove that, if $\operatorname{sur}(\operatorname{dom} \chi_S) \neq \emptyset$, then the nine sets $\overline{D(S)}$, $\overline{\operatorname{co} D(S)}$, $\overline{\operatorname{dom} \chi_S}$, $\overline{\operatorname{int} D(S)}$, $\overline{\operatorname{int}(\operatorname{co} D(S))}$, $\overline{\operatorname{int}(\operatorname{dom} \chi_S)}$, $\overline{\operatorname{sur} D(S)}$, $\overline{\operatorname{sur}(\operatorname{co} D(S))}$ and $\overline{\operatorname{sur}(\operatorname{dom} \chi_S)}$ coincide. ("Sur" is defined in Section 12.) These results strengthen results of Rockafellar that $\operatorname{int} D(S)$ is convex and that, if $\operatorname{int}(\operatorname{co} D(S)) \neq \emptyset$ then $\overline{D(S)}$ is convex, and also settle in the affirmative an open problem as to whether an absorbing point of $D(S)$ is necessarily an interior point. We do not know if the results analogous to the six set theorem and the nine set theorem hold with "χ_S" replaced by "ψ_S". We end Section 18 by specializing to the reflexive case, in which the answer to the above question is in the affirmative. In Section 19, we introduce a convex function ξ_S "dual" to ψ_S and show the connection between this function and Brézis–Haraux appproximation, which is concerned with finding conditions under which $R(S_1 + S_2)$ is "approximately equal" to $R(S_1) + R(S_2)$ in the sense that

$$\left.\begin{array}{c}\overline{R(S_1 + S_2)} = \overline{R(S_1) + R(S_2)} \\ \text{and} \\ \operatorname{int}\big[R(S_1 + S_2)\big] = \operatorname{int}\big[R(S_1) + R(S_2)\big].\end{array}\right\} \tag{19.0.1}$$

Brézis–Haraux give two conditions ((19.0.2) and (19.0.3)) which imply (19.0.1). We show that each of these imply that

$$R(S_1) + R(S_2) \subset \operatorname{dom} \xi_{S_1 + S_2}, \tag{19.3.1}$$

and that (19.3.1) in turn implies (19.0.1). There is a third condition, due to Pazy, which implies (19.3.1). We do not know if Pazy's condition implies (19.3.1). (See Problem 19.6.) The next flowchart shows which sections are needed for an understanding of Chapter IV.



Chapter V is motivated by the following result of Rockafellar and recent generalizations of it. *If E is reflexive, S_1 and S_2 are maximal monotone and*

$$D(S_1) \cap \operatorname{int} D(S_2) \neq \emptyset \tag{20.0.1}$$

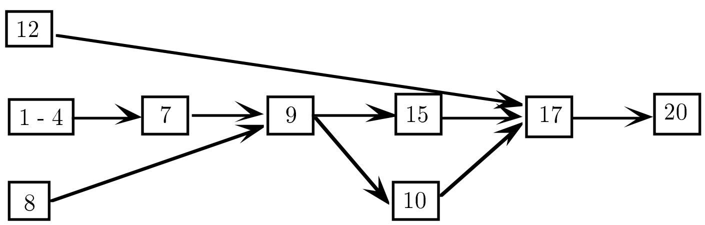
then $S_1 + S_2$ is maximal monotone. Indeed, we give a proof of this result in Theorem 20.5. Apart from this, Section 20 is devoted to setting up the machinery that we shall need for the more general results that appear later on in the chapter. Much of the discussion centers on the "γ–condition" (20.1.1). We shall explain in the text why this ugly condition is, in fact, forced on us by the nature of our problem. We then use the minimax technique to prove in Lemma 20.1 the equivalence of the γ–condition with condition (20.1.2), which says that there exist $x_1^*,\ x_2^* \in E^*$ and $z \in E$ such that, for all $(s_1, s_1^*) \in G(S_1)$ and $(s_2, s_2^*) \in G(S_2)$,

$$2\langle s_1 - z, s_1^* - x_1^*\rangle + 2\langle s_2 - z, s_2^* - x_2^*\rangle \geq \|z\|^2 + \|x_1^* + x_2^*\|^2 + 2\langle z, x_1^* + x_2^*\rangle.$$

An argument due originally to Minty and Browder enables us to deduce in Lemma 20.2 that there exists $(z, z^*) \in G(S_1 + S_2)$ such that

$$\|z\|^2 + \|z^*\|^2 + 2\langle z, z^*\rangle = 0.$$

Using an extension of the local boundedness theorem established in Lemma 17.2(a), we then prove in Lemma 20.3 that Rockafellar's condition, (20.0.1), implies the γ–condition, and so all the results above hold. Our final step towards Theorem 20.5 is Lemma 20.4, in which we boostrap by translating first in E and then in E^*. Theorem 20.5 itself is proved by using the criterion for maximality in Theorem 10.3. The next flowchart shows which sections are needed for an understanding of Section 20.

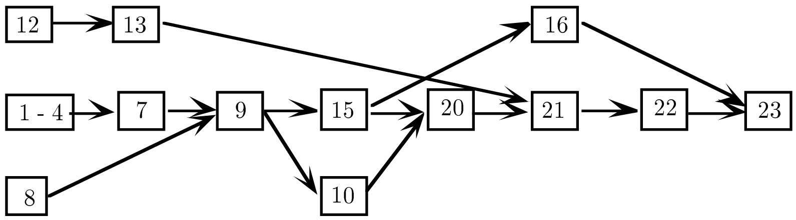


Section 21 is about the "dom–dom constraint qualification",

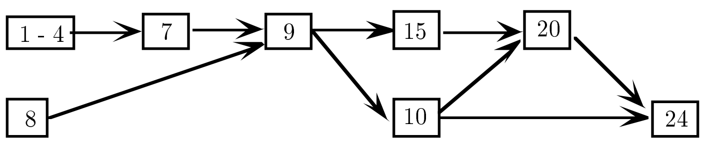
$$\operatorname{dom} \chi_{S_1} - \operatorname{dom} \chi_{S_2} \quad \text{is absorbing.} \tag{21.0.1}$$

Indeed, we shall prove in Theorem 21.3 that the maximal monotonicity of $S_1 + S_2$ remains true even if (20.0.1) is weakened to (21.0.1). This result is established using exactly the same steps as in Section 20, except that Lemma 21.1 is much harder than the result to which it corresponds, Lemma 20.3. (Lemma 21.1 uses the dom–dom lemma, Lemma 13.1 rather than the dom lemma, Lemma 12.2.) Actually, our analysis contains a necessary and sufficient condition (still, for reflexive spaces) for the sum of maximal monotone multifunctions to be maximal monotone. We have set this out in Theorem 21.4, but we suspect that it may be too complicated to be of any practical use.

In Section 22, we prove that the six set theorem and the nine set theorem established in Section 18 for $D(S)$ (S maximal monotone on a general Banach space) have analogs for $D(S_1) - D(S_2)$ (S_1 and S_2 maximal monotone on a reflexive Banach space). We deduce from this in Section 23 the equivalence of several constraint qualifications that have been proposed recently. The next flowchart shows which sections are needed for an understanding of Sections 21—23.



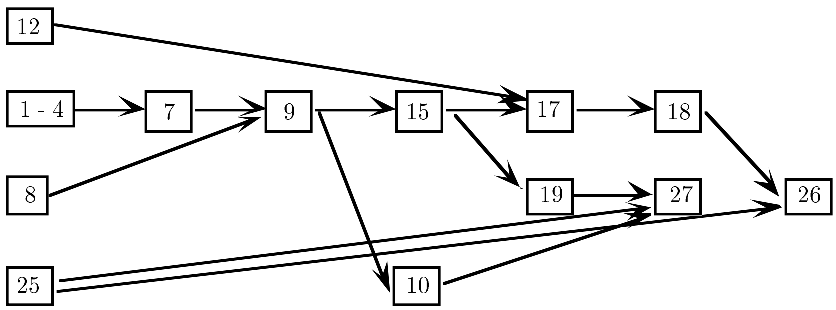
We close the chapter by showing in Section 24 how the techniques of these notes can be used to establish the Brézis–Crandall–Pazy result on the maximal monotonicity of the sum. The final flowchart for Chapter V shows which sections are needed for an understanding of Section 24. (The only part of Section 20 that is needed is Lemma 20.2, which does not depend on Baire's theorem.)



In Chapter VI, we discuss some of the subclasses of the class of maximal monotone multifunctions that have been introduced over the years. In Section 25, we define and give the basic properties of those that are of "type (D)", "type (FP)", "type (FPV)", "type (NI)", "type (ANA)", and those that are "strongly maximal monotone".

Now suppose that E is nonreflexive and S is maximal monotone. It has been noted in Problem 18.9 that we do not know if $\overline{D(S)}$ is necessarily convex, and it is also noted in the remarks preceding Definition 25.4 that we do not know if S is necessarily of type (FPV). The connection between these observations is clarified in Section 26. Indeed, we prove in Theorem 26.3 that if $\overline{D(S)} \neq \overline{\operatorname{dom}\psi_S}$ (in particular, if $\overline{D(S)}$ is not convex) then S is not of type (FPV), and in Theorem 26.1 that if S is not of type (FPV) then we have a negative answer to the long–standing question whether Rockafellar's sum theorem is true in nonreflexive spaces. In Section 27, we return to our consideration of the function ξ_S introduced in Section 19, and show that in the three cases where it has been proved that $\overline{R(S)}$ is convex one can, in fact, prove that $\overline{R(S)} = \overline{\operatorname{dom}\xi_S}$. Some of the results of Section 27 are "dual" to those of Section 26, while others seem to rely on totally different principles.

The next flowchart shows which sections are needed for an understanding of Chapter VI. (The only part of Section 19 needed in Section 27 is the definition of ξ_S.)

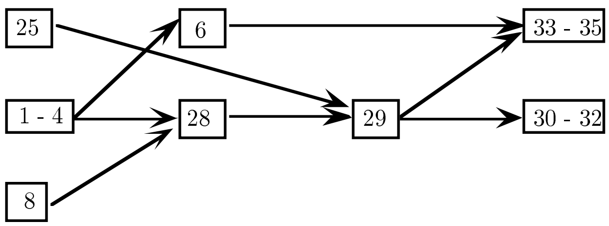


In Chapter VII, we consider the properties of subdifferentials. There are various ways of approaching this topic. In the end, the easiest seems to be through the formula for the subdifferential of the sum of two convex functions, which we consider in Section 28. The main work for this is actually in Lemma 28.1 (a form of the Fenchel duality theorem in which we do not explicitly mention conjugates). We establish this using the minimax technique and a "scaling" argument, while the formula for the subdifferential of the sum is obtained by a simple bootstrapping procedure in Theorem 28.2. In Section 29, we use the minimax technique and Ekeland's variational principle to obtain a version of the Brøndsted–Rockafellar theorem, which we combine with the formula for the subdifferential of a sum to obtain our main result on the existence of subgradients, Theorem 29.4. With one significant exception, all the later results on subdifferentials depend on this theorem. In Corollary 29.5, we obtain (modulo some simple bootstrapping) Rockafellar's classical result that subdifferentials are maximal monotone. The main result in Section 30 is Theorem 30.3, in which we establish that subdifferentials are of type (FP), and the main result in Section 31 is Theorem 31.3, in which we establish that subdifferentials are of type (FPV). In Section 32, we prove that sudifferentials are strongly maximal monotone. This is the case for which Theorem 29.4 does not seem to be adequate. In order to handle this, we prove in Lemma 32.1 a generalization of Corollary 29.5 in which $\{0\}$ is replaced by any nonempty $w(E, E^*)$–compact convex subset of E.

In Section 35, we shall prove that subdifferentials are maximal monotone of type (D). In fact, we shall define a slightly stronger property ("type (DS)") and prove in Theorem 35.3 that subdifferentials enjoy this stronger property. This implies, in particular, that subdifferentials are of "dense type" in the sense of Gossez. Gossez's analysis relies on the theory of locally convex spaces. We use instead some properties of the biconjugates of convex functions. The preliminary work for Theorem 35.3 is contained in Sections 33 and 34. The main result of Section 33 is Theorem 33.3, in which we establish the formula for the biconjugate of the pointwise maximum of a finite number of functions.

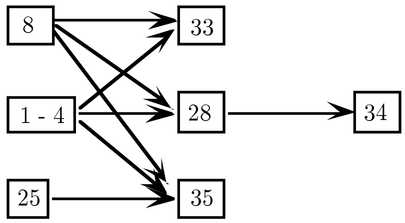
What is curious is that we can establish this result in a situation in which we do not have a simple explicit formula for the *conjugate* of the pointwise maximum. In Section 34, we will define a new topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ on E^{**} which lies between the weak* topology $w(E^{**}, E^*)$ and the norm topology $\mathcal{T}_{\| \|}(E^{**})$. The main result here is Theorem 34.7, in which we use Theorem 33.3 to show that if f is a somewhere finite convex lower semicontinuous function on E then the canonical image of $G(\partial f)$ is dense in $G(\partial f^*)$ in the topology $\mathcal{T}_{\| \|}(E^*) \times \mathcal{T}_{\mathcal{CLB}}(E^{**})$.

The next flowchart shows which sections are needed for an understanding of Chapter VII.



However, readers already familiar with the formula for the subdifferential of a sum, the Brøndsted–Rockafellar theorem and the elementary properties of the Fenchel conjugate should be able to read most of Chapter VII without needing to refer to the previous chapters of these notes.

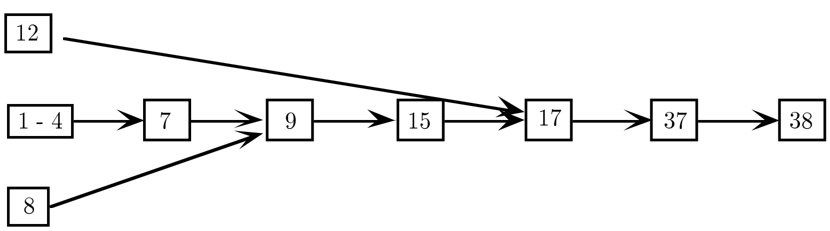
We give a brief discussion of some of the properties of (possibly unbounded) positive *linear* operators in Chapter VIII. Theorem 36.2 contains a necessary and sufficient condition for a positive linear operator to be maximal monotone. Theorem 37.1 gives a sufficient condition for the sum of two maximal monotone linear operators to be maximal monotone. We shall see that, for linear operators, Rockafellar's original sum theorem is true *even without the assumption of reflexivity*. In Section 38, we consider which positive linear operators belong to various subclasses of the family of maximal monotone multifunctions introduced in Section 25: we use the minimax technique to prove in Theorem 38.2 that every maximal monotone linear operator is also of type (FPV), in Theorem 38.3 that every linear operator that is maximal monotone of type (NI) is also of type (FP), in Theorem 38.5 that every maximal monotone linear operator is also strongly maximal monotone, and in Theorem 38.6 that every continuous positive linear operator is of type (ANA). The next flowchart shows which sections are needed for an understanding of Chapter VIII.

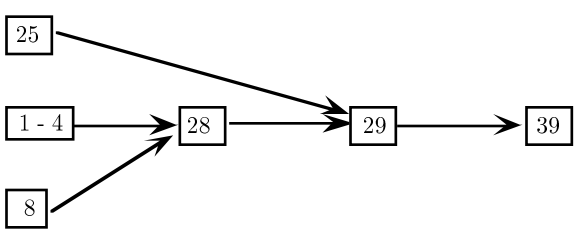


In Chapter IX, we shall give some cases in which Rockafellar's sum theorem (see the discussion of Chapter V above) is true in the nonreflexive case.

- We shall prove in Theorem 40.4 the result of Heisler that *if* $S_1\colon\ E \mapsto 2^{E^*}$ *and* $S_2\colon\ E \mapsto 2^{E^*}$ *are maximal monotone and* $D(S_1) = D(S_2) = E$ *then* $S_1 + S_2$ *is maximal monotone.*
- We shall prove in Theorem 41.2 that *if* C *is a nonempty closed convex subset of* E, $S\colon\ E \mapsto 2^{E^*}$ *is maximal monotone and* $\operatorname{int} D(S) \supset C$ *then* $S+N_C$ *is maximal monotone.* (Here N_C is the *normality multifunction* associated with the set C, defined in Section 8.)
- We shall prove in Theorem 41.6 that *if* C *is a closed convex subset of* E, $D(S)$ *is a subspace of* E, $S\colon\ D(S) \mapsto E^*$ *is linear and maximal monotone and* $D(S) \cap \operatorname{int} C \neq \emptyset$ *then* $S + N_C$ *is maximal monotone.*
- We shall prove in Theorems 42.1 and 42.2 the two results of Bauschke that *if* $S\colon\ E \mapsto 2^{E^*}$ *is maximal monotone and* $T\colon\ E \mapsto E^*$ *is skew and linear then* $S + T$ *is maximal monotone* and *if* f *is a somewhere finite convex lower semicontinuous function on* E *and* $T\colon\ E \mapsto E^*$ *is positive and linear then* $\partial f + T$ *is maximal monotone.*

The last two flowcharts show which sections are needed for an understanding of Chapter IX.





In the final chapter, Chapter X, we collect together some of the open problems that have appeared in the body of the text.