Techniques for maximal monotonicity

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Introduction

The purpose of this paper is to describe three techniques that are useful for the investigation of monotone sets and multifunctions, and give two applications of them.

The three techniques use the "fg–theorem", the "big convexification" of a subset of $E \times E^*$ or a multifunction $E \mapsto 2^{E^*}$ (we suppose throughout that E is a nontrivial real Banach space with dual E^*), and the "convex function associated with a multifunction $E \mapsto 2^{E^*}$". The applications that we shall give will be the derivation of various criteria for monotone subset of $E \times E^*$ to be maximal monotone in the special case where E is reflexive, and a slight generalization of Rockafellar's theorem on the maximal monotonicity of the sum of maximal monotone multifunctions on a reflexive space. Remark 27 at the end of the paper contains pointers to much stronger results that can be proved using these techniques.

We use a combination of the one–dimensional Hahn–Banach theorem, the Banach–Alaoglu theorem and a minimax theorem to establish the *fg–theorem*, Theorem 3. Though monotonicity is not mentioned in it, the fg–theorem is, in fact, an abstraction of results on monotonicity that appeared in our paper [11]. Thus this result is a bridge between functional analysis and monotonicity.

The *big convexification of a subset G of $E \times E^*$* is a way of embedding G in a large convex set in such a way that monotone sets can be characterized by a different kind of inequality. The details of this are given in the "*pqr*–lemma", Lemma 4.

We will give the definition of the *convex function χ_S associated with the multifunction S* in Definition 21.

In Lemma 7, we use the fg–theorem to obtain an equivalence valid for any nonempty subset of $E \times E^*$ when E is reflexive. We apply this equivalence to prove in Theorem 13 that *if M is a monotone subset of $E \times E^*$ then*

$$M \text{ is maximal monotone} \iff M + G(-J) = E \times E^*,$$

where J is the duality map, and we deduce in Theorem 14 that if $S\colon E \mapsto 2^{E^*}$ is maximal monotone then $R(S + J) = E^*$. This is one direction of Rockafellar's "surjectivity theorem". We note, however, that Theorems 13 and 14 hold for *any* reflexive space, not only for ones where the norms of E and E^* are strictly convex.

In Lemma 16, we use the fg–theorem to obtain an equivalence valid for any pair of nontrivial multifunctions $E \mapsto 2^{E^*}$ when E is reflexive. (In a certain sense, Lemma 16 is a two–dimensional form of Lemma 7, and its consequence, Lemma 17, is a two–dimensional form of the consequence Lemma 8(b) of Lemma 7.) We apply this equivalence to prove in Theorem 26 the following slight extension of Rockafellar's sum theorem: *Let E be reflexive, $S_1\colon E \mapsto 2^{E^*}$ and $S_2\colon E \mapsto 2^{E^*}$ be maximal monotone and* $\operatorname{co} D(S_1) \cap \operatorname{int} \operatorname{co} D(S_2) \neq \emptyset$. *Then $S_1 + S_2$ is maximal monotone.*

Between Lemma 16 and Theorem 26, we use Baire's theorem to prove in Lemma 19 a result on convex functions that we will apply in Lemma 23 to the convex functions

χ_S to obtain a generalization (to not necessarily monotone multifunctions) of the local boundedness theorem for monotone multifunctions. Lemma 23 leads to Lemma 24, which is the other critical part of our proof of Theorem 26.

We remark finally that the combination of the one–dimensional Hahn–Banach theorem, the Banach–Alaoglu theorem and a minimax theorem that we use here to establish the fg–theorem is particularly well adapted to problems on the existence of linear functionals, both in relation to monotonicity and in other cases too. Many more examples of this "minimax technique" can be found in [12].

The first technique: the fg–theorem

The main result of this section is the fg–theorem, Theorem 3. This result is interesting in that is uses the minimax theorem twice, first in a scalar form to produce the bound β, and then in a vector form in which this bound is used to establish compactness. It is also worthy of note that the bound β is produced without the use of Baire's theorem.

We shall use the following minimax theorem, which follows from a result of Fan (see [4]). (See also the paper [6] by König and our paper [10] for simple generalizations of Fan's result.) It is important that the set A not be required to have a topological structure.

Theorem 1. *Let A be a nonempty convex subset of a vector space, B be a nonempty convex subset of a vector space and B also be a compact Hausdorff topological space. Let $h\colon\ A \times B \mapsto \mathbb{R}$ be convex on A, and concave and upper semicontinuous on B. Then*

$$\inf_A \max_B h = \max_B \inf_A h.$$

Before proving the fg–theorem, we give a simple result that we shall call the "perfect square trick".

Lemma 2. *If $x \in E$ and $x^* \in E^*$ then $\|x\|^2 + \|x^*\|^2 + 2\langle x, x^*\rangle \geq 0$.*

Proof. $\|x\|^2 + \|x^*\|^2 + 2\langle x, x^*\rangle \geq \|x\|^2 + \|x^*\|^2 - 2\|x\|\|x^*\| = (\|x\| - \|x^*\|)^2$. ∎

Theorem 3. *Let A be a nonempty convex subset of a vector space, F be a Banach space, $f\colon\ A \mapsto \mathbb{R}$ be convex and $g\colon\ A \mapsto F$ be affine. Then* (3.1) $\Longleftrightarrow$ (3.2).

$$a \in A \quad\Longrightarrow\quad f(a) + \|g(a)\|^2 \geq 0. \tag{3.1}$$

$$\left.\begin{array}{l}\textit{There exists } y^* \in F^* \textit{ such that}\\ \quad a \in A \quad\Longrightarrow\quad f(a) - 2\langle g(a), y^*\rangle - \|y^*\|^2 \geq 0.\end{array}\right\} \tag{3.2}$$

Proof. ($\Longrightarrow$) Let $a_0 \in A$, and n be an integer such that $n \geq \|g(a_0)\|$. Put

$$A_n := \{a \in A\colon\ \|g(a)\| \leq n\}.$$

Since $a_0 \in A_n$, A_n is not empty, and A_n is clearly convex. Define $h\colon\ A \times [0,\infty) \mapsto \mathrm{I\!R}$ by

$$h(a,\beta) := f(a) + 2\beta\|g(a)\| - \beta^2.$$

Using (3.1), we have

$$\inf_{A_n} \max_{[0,n]} h \ge \inf_{a\in A_n} h(a, \|g(a)\|) \ge 0.$$

The function h is convex on A_n, and concave and continuous on $[0,n]$. Since $[0,n]$ is compact, from the minimax theorem, Theorem 1,

$$\max_{[0,n]} \inf_{A_n} h \ge 0,$$

from which

$$C_n := \bigcap_{a\in A_n} \{\beta \in [0,\infty)\colon\ h(a,\beta) \ge 0\} \ne \emptyset.$$

C_n is clearly compact. Further, the sets C_n decrease as n increases. Consequently,

$$\bigcap_{n\ge\|g(a_0)\|} C_n \ne \emptyset.$$

Since

$$A = \bigcup_{n\ge\|g(a_0)\|} A_n,$$

it now follows that

$$\text{there exists } \beta \ge 0 \text{ such that} \quad a \in A \quad \Longrightarrow \quad f(a) + 2\beta\|g(a)\| - \beta^2 \ge 0. \tag{3.3}$$

Now define $h\colon\ A \times F^* \mapsto \mathrm{I\!R}$ by

$$h(a,y^*) := f(a) - 2\langle g(a), y^*\rangle - \|y^*\|^2.$$

Let $B := \{y^* \in F^*\colon\ \|y^*\| \le \beta\}$, with the topology $w(F^*,F)$. From the Banach–Alaoglu theorem, B is compact. Let $a \in A$. From the one–dimensional Hahn–Banach theorem, we can find $y^* \in B$ such that $\langle g(a), y^*\rangle = -\beta\|g(a)\|$. From (3.3),

$$h(a,y^*) = f(a) - 2\langle g(a), y^*\rangle - \|y^*\|^2 \ge f(a) + 2\beta\|g(a)\| - \beta^2 \ge 0.$$

Thus

$$\inf_{A} \max_{B} h \ge 0.$$

Since A and B are convex, and h is convex on A and concave and upper semicontinuous on B, from the minimax theorem, Theorem 1,

$$\max_{B} \inf_{A} h \ge 0,$$

which gives (3.2).

($\Longleftarrow$) From the perfect square trick, Lemma 2, for all $a \in A$ and $y^* \in F^*$,

$$\|g(a)\|^2 \geq -2\langle g(a), y^*\rangle - \|y^*\|^2. \blacksquare$$

The second technique: the "big convexification"

Much of this section first appeared (with a different notation) in the paper [3] by Coodey–Simons. We write $\mathbb{R}^{(E\times E^*)}$ for the direct sum of $E \times E^*$ copies of $\mathbb{R}$, that is the set of functions μ: $E \times E^* \mapsto \mathbb{R}$ such that

$$\{(s, s^*) \in E \times E^*\colon\ \mu(s, s^*) \neq 0\} \quad \text{is finite.}$$

$\mathbb{R}^{(E\times E^*)}$ is a vector space. If $(y, y^*) \in E \times E^*$ then $\delta_{(y,y^*)} \in \mathbb{R}^{(E\times E^*)}$, where $\delta_{(y,y^*)}$ is defined by

$$\delta_{(y,y^*)}(s, s^*) := \begin{cases} 1 & \text{if } (s, s^*) = (y, y^*); \\ 0 & \text{otherwise.} \end{cases}$$

We can also think of $\mathbb{R}^{(E\times E^*)}$ as the set of signed measures on $E \times E^*$ with finite support. Then $\delta_{(y,y^*)}$ corresponds to the point mass at (y, y^*). Now suppose that $\emptyset \neq G \subset E \times E^*$. Let $\mathcal{CO}(G)$ be the convex hull in $\mathbb{R}^{(E\times E^*)}$ of $\{\delta_{(y,y^*)}\colon\ (y, y^*) \in G\}$. Explicitly, if $\mu \in \mathbb{R}^{(E\times E^*)}$ then $\mu \in \mathcal{CO}(G)$ if, and only if

$$\mu \geq 0 \text{ on } E \times E^*, \quad \mu(s, s^*) > 0 \Longrightarrow (s, s^*) \in G \quad \text{and} \quad \sum_{(s,s^*)\in G} \mu(s, s^*) = 1.$$

$\mathcal{CO}(G)$ is the "big convexification" of G. Continuing the "measure theory" analogy introduced above, we can think of $\mathcal{CO}(G)$ as the set of probability measures on $E \times E^*$ with finite support contained in G.

We now define the three linear operators p: $\mathbb{R}^{(E\times E^*)} \mapsto E$, q: $\mathbb{R}^{(E\times E^*)} \mapsto E^*$ and r: $\mathbb{R}^{(E\times E^*)} \mapsto \mathbb{R}$ by

$$p(\mu) := \textstyle\sum_{(s,s^*)\in E\times E^*} \mu(s, s^*)s,$$
$$q(\mu) := \textstyle\sum_{(s,s^*)\in E\times E^*} \mu(s, s^*)s^*$$

and

$$r(\mu) := \textstyle\sum_{(s,s^*)\in E\times E^*} \mu(s, s^*)\langle s, s^*\rangle.$$

We will use the fact that, for all $(y, y^*) \in E \times E^*$,

$$p(\delta_{(y,y^*)}) = y, \quad q(\delta_{(y,y^*)}) = y^* \quad \text{and} \quad r(\delta_{(y,y^*)}) = \langle y, y^*\rangle.$$

A subset M of $E \times E^*$ is *monotone* if

$$(x, x^*) \text{ and } (y, y^*) \in M \quad \Longrightarrow \quad \langle x - y, x^* - y^*\rangle \geq 0.$$

In the pqr–lemma, Lemma 4, we give a characterization of the monotone subsets of $E \times E^*$ in terms of p, q and r. The manipulations contained in Lemma 4 are part of the folklore of monotonicity.

Lemma 4. *Let M be a nonempty subset of $E \times E^*$. Then M is monotone $\iff$*

$$\mu \in \mathcal{CO}(M) \quad \Longrightarrow \quad r(\mu) \ge \langle p(\mu), q(\mu)\rangle.$$

Proof. Suppose first that M is a monotone subset of $E \times E^*$. Let $(s_1, s_1^*), \ldots, (s_m, s_m^*)$ be an enumeration of those elements (s, s^*) of M for which $\mu(s, s^*) > 0$, and write α_i for $\mu(s_i, s_i^*)$. Then, with the summations going from 1 to m,

$$\begin{aligned} r(\mu) - \langle p(\mu), q(\mu)\rangle &= \textstyle\sum_i \alpha_i \langle s_i, s_i^*\rangle - \langle \sum_i \alpha_i s_i, \sum_i \alpha_i s_i^*\rangle \\ &= \textstyle\sum_{i,j} \alpha_i\alpha_j \langle s_i, s_i^*\rangle - \sum_{i,j} \alpha_i\alpha_j \langle s_i, s_j^*\rangle \\ &= \textstyle\sum_{i,j} \alpha_i\alpha_j \langle s_i, s_i^* - s_j^*\rangle \\ &= \textstyle\sum_{i<j} \alpha_i\alpha_j \langle s_i, s_i^* - s_j^*\rangle + \sum_{j<i} \alpha_i\alpha_j \langle s_i, s_i^* - s_j^*\rangle \\ &= \textstyle\sum_{i<j} \alpha_i\alpha_j \langle s_i, s_i^* - s_j^*\rangle + \sum_{i<j} \alpha_i\alpha_j \langle s_j, s_j^* - s_i^*\rangle \\ &= \textstyle\sum_{i<j} \alpha_i\alpha_j \langle s_i - s_j, s_i^* - s_j^*\rangle \ge 0. \end{aligned}$$

We leave the proof of the converse to the reader. ▌

We will actually use the following corollary of the *pqr*–lemma:

Corollary 5. *(a) If M is a nonempty monotone subset of $E \times E^*$ then*

$$\mu \in \mathcal{CO}(M) \quad \Longrightarrow \quad 2r(\mu) + \|p(\mu)\|^2 + \|q(\mu)\|^2 \ge 0. \tag{5.1}$$

(b) If M_1 and M_2 are nonempty monotone subsets of $E \times E^*$ then

$$\left.\begin{aligned} &(\mu_1, \mu_2) \in \mathcal{CO}(M_1) \times \mathcal{CO}(M_2) \quad \Longrightarrow \\ &\qquad 2r(\mu_1 + \mu_2) + 2\|p(\mu_2 - \mu_1)\|\|q(\mu_2)\| \\ &\qquad\qquad + \|p(\mu_1)\|^2 + \|q(\mu_1 + \mu_2)\|^2 \ge 0. \end{aligned}\right\} \tag{5.2}$$

Proof. (a) Using the *pqr*–lemma, Lemma 4, and the perfect square trick, Lemma 2,

$$2r(\mu) + \|p(\mu)\|^2 + \|q(\mu)\|^2 \ge \|p(\mu)\|^2 + \|q(\mu)\|^2 + 2\langle p(\mu), q(\mu)\rangle \ge 0.$$

(b) Using arguments similar to those above,

$$\begin{aligned} &2r(\mu_1 + \mu_2) + \|p(\mu_1)\|^2 + \|q(\mu_1 + \mu_2)\|^2 \\ &\qquad \ge 2r(\mu_1) + 2r(\mu_2) - 2\langle p(\mu_1), q(\mu_1 + \mu_2)\rangle \\ &\qquad \ge 2\langle p(\mu_1), q(\mu_1)\rangle + 2\langle p(\mu_2), q(\mu_2)\rangle - 2\langle p(\mu_1), q(\mu_1 + \mu_2)\rangle \\ &\qquad = 2\langle p(\mu_2 - \mu_1), q(\mu_2)\rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} &2r(\mu_1 + \mu_2) + 2\|p(\mu_2 - \mu_1)\|\|q(\mu_2)\| + \|p(\mu_1)\|^2 + \|q(\mu_1 + \mu_2)\|^2 \\ &\qquad \ge 2\|p(\mu_2 - \mu_1)\|\|q(\mu_2)\| + 2\langle p(\mu_2 - \mu_1), q(\mu_2)\rangle \ge 0. \end{aligned}$$ ▌

The subset M of $E \times E^*$ is said to be *maximal monotone* if M is monotone, and M has no proper monotone extension. This is equivalent to the statement:

$$(z, z^*) \in E \times E^* \text{ and } \inf_{(s,s^*) \in M} \langle s - z, s^* - z^*\rangle \ge 0 \quad \Longrightarrow \quad (z, z^*) \in M.$$

In Lemma 17, we will need the more quantitative version of this that is provided by Lemma 6. (a) follows by taking $(s, s^*) := (z, z^*)$, (b) is immediate from the definition of maximal monotonicity, and (c) follows from (a) and (b).

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Lemma 6. *Let $M \subset E \times E^*$ be maximal monotone.*
(a) If $(z, z^*) \in M$ then $\inf_{(s,s^*)\in M}\langle s - z, s^* - z^*\rangle = 0$.
(b) If $(z, z^*) \in E \times E^* \setminus M$ then $\inf_{(s,s^*)\in M}\langle s - z, s^* - z^*\rangle < 0$.
(c) For all $(z, z^*) \in E \times E^*$, $\inf_{(s,s^*)\in M}\langle s - z, s^* - z^*\rangle \le 0$.

Criteria for maximal monotonicity in reflexive spaces

In this section (apart from in Remark 11), we assume that E is reflexive. The main result is the "perfect square" criterion for maximality, Theorem 9, in which we prove that *if M is a monotone subset of $E \times E^*$, then M is maximal monotone if, and only if, for all $(w, w^*) \in E \times E^*$, there exists $(x, x^*) \in M$ such that*

$$\|x - w\|^2 + \|x^* - w^*\|^2 + 2\langle x - w, x^* - w^*\rangle = 0.$$

We start by using the fg–theorem, Theorem 3, to prove an equivalence for *arbitrary* nonempty subsets of $E \times E^*$. We note that (7.1) is identical with (5.1), except that M has been replaced by G. The analysis in this section is derived from results that first appeared in our paper [11], however the fg–theorem enables us to give much simpler proofs. If $x \in E$, we write $\widehat{x}$ for the canonical image of x in the bidual of E.

Lemma 7. *Let E be reflexive and $\emptyset \ne G \subset E \times E^*$. Then the conditions* (7.1) *and* (7.2) *are equivalent:*

$$\mu \in \mathcal{CO}(G) \quad\Longrightarrow\quad 2r(\mu) + \|p(\mu)\|^2 + \|q(\mu)\|^2 \ge 0. \tag{7.1}$$

$$\left.\begin{array}{l}\textit{There exists } (x, x^*) \in E \times E^* \textit{ such that}\\ (s, s^*) \in G \quad\Longrightarrow\quad 2\langle s - x, s^* - x^*\rangle \ge \|x^*\|^2 + \|x\|^2 + 2\langle x, x^*\rangle.\end{array}\right\} \tag{7.2}$$

Proof. We shall establish the equivalence of (7.1) and (7.2) by proving their equivalence with the intermediate conditions (7.3) — (7.5) below:

$$\left.\begin{array}{l}\text{There exists } y^* \in (E \times E^*)^* \text{ such that}\quad \mu \in \mathcal{CO}(G) \Longrightarrow\\ \quad 2r(\mu) - 2\langle (p(\mu), q(\mu)), y^*\rangle - \|y^*\|^2 \ge 0.\end{array}\right\} \tag{7.3}$$

$$\left.\begin{array}{l}\text{There exists } (x, x^*) \in E \times E^* \text{ such that}\quad \mu \in \mathcal{CO}(G) \Longrightarrow\\ \quad 2r(\mu) - 2\langle p(\mu), x^*\rangle - 2\langle x, q(\mu)\rangle - \|x\|^2 - \|x^*\|^2 \ge 0.\end{array}\right\} \tag{7.4}$$

$$\left.\begin{array}{l}\text{There exists } (x, x^*) \in E \times E^* \text{ such that}\quad (s, s^*) \in G \Longrightarrow\\ \quad 2\langle s, s^*\rangle - 2\langle s, x^*\rangle - 2\langle x, s^*\rangle - \|x\|^2 - \|x^*\|^2 \ge 0.\end{array}\right\} \tag{7.5}$$

((7.1)$\Longleftrightarrow$(7.3)) This follows from the fg–theorem, Theorem 3, with $F := E\times E^*$, $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$, $A := \mathcal{CO}(G)$ and, for all $\mu \in A$,

$$f(\mu) := 2r(\mu) \quad\text{and}\quad g(\mu) := \big(p(\mu), q(\mu)\big).$$

((7.3)$\Longleftrightarrow$(7.4)) This is clear since any element y^* of F^* can be written in the form $(x^*, \widehat{x})$ for some $(x, x^*) \in E \times E^*$, and $\|y^*\| = \sqrt{\|x\|^2 + \|x^*\|^2}$.

((7.4)$\iff$(7.5)) If (7.4) is satisfied then (7.5) follows by restricting μ to the values $\delta_{(s,s^*)}$. If, conversely, (7.5) is satisfied and $\mu \in \mathcal{CO}(G)$ then (7.4) follows by multiplying the left hand side of the inequality in (7.5) by $\mu(s,s^*)$ and summing up over all $(s,s^*) \in G$.

((7.5) $\iff$ (7.2)) This can be seen by rearranging the terms and adding $\pm 2\langle x, x^*\rangle$ to each side.

This completes the proof of Lemma 7. ▮

Note that we introduce the term $2\langle x, x^*\rangle$ *after* the minimax theorem has been used in the proof that (7.1) implies (7.2) above, since this term is not generally semicontinuous on $E \times E^*$ with respect to the product of the topologies $w(E,E^*)$ and $w(E^*,E)$.

Lemma 8. *Let E be reflexive and M be a nonempty monotone subset of $E \times E^*$. Then:*
(a) There exists $(x,x^*) \in E \times E^*$ such that

$$(s,s^*) \in M \quad \Longrightarrow \quad 2\langle s-x, s^*-x^*\rangle \geq \|x^*\|^2 + \|x\|^2 + 2\langle x, x^*\rangle. \tag{8.1}$$

(b) Suppose now that M is a maximal monotone *subset of $E \times E^*$. Then there exists $(x,x^*) \in M$ such that*

$$\|x\|^2 + \|x^*\|^2 + 2\langle x, x^*\rangle = 0.$$

Proof. We note that (8.1) is identical with (7.2), except that G has been replaced by M. Thus (a) immediate from Corollary 5(a) and Lemma 7. Now let $(x,x^*) \in E \times E^*$ be as in (a). From the perfect square trick, Lemma 2,

$$(s,s^*) \in M \quad \Longrightarrow \quad 2\langle s-x, s^*-x^*\rangle \geq 0.$$

Since M is *maximal* monotone, $(x,x^*) \in M$, and (b) then follows from the perfect square trick again by substituting $(s,s^*) := (x,x^*)$ in (8.1). ▮

We now come to the perfect square criterion for maximality.

Theorem 9. *Let E be reflexive and M be a monotone subset of $E \times E^*$. Then M is maximal monotone $\iff$ for all $(w,w^*) \in E \times E^*$, there exists $(x,x^*) \in M$ such that*

$$\|x-w\|^2 + \|x^*-w^*\|^2 + 2\langle x-w, x^*-w^*\rangle = 0. \tag{9.1}$$

Proof. ($\Longrightarrow$) We apply Lemma 8(b), with M replaced by $M - (w,w^*) \subset E \times E^*$, which is also maximal monotone.

($\Longleftarrow$) Let $(w,w^*) \in E \times E^*$ and

$$(x,x^*) \in M \quad \Longrightarrow \quad \langle x-w, x^*-w^*\rangle \geq 0.$$

Choose $(x,x^*) \in M$ as in (9.1). Then

$$\|x-w\|^2 + \|x^*-w^*\|^2 \leq 0,$$

and so $(w,w^*) = (x,x^*) \in M$. Thus M is maximal monotone. ▮

We next deduce from Theorem 9 the "negative alignment" criterion for maximal monotonicity.

Corollary 10. *Let E be reflexive and M be a monotone subset of $E \times E^*$. Then M is maximal monotone $\iff$ for all $(w, w^*) \in E \times E^* \setminus M$, there exists $(x, x^*) \in M$ such that*

$$x \neq w,\ x^* \neq w^* \quad \text{and} \quad \langle x - w, x^* - w^* \rangle = -\|x - w\|\|x^* - w^*\|. \tag{10.1}$$

Proof. ($\Longrightarrow$) Let $(w, w^*) \in E \times E^* \setminus M$. Choose $(x, x^*) \in M$ as in (9.1). Clearly, either $x \neq w$ or $x^* \neq w^*$ (or both!). It follows from the proof of the perfect square trick, Lemma 2, that $\|x - w\| = \|x^* - w^*\|$. So, in fact $x \neq w$ *and* $x^* \neq w^*$. Using (9.1) and the proof of the perfect square trick again,

$$\begin{aligned} 0 &= \|x - w\|^2 + \|x^* - w^*\|^2 + 2\langle x - w, x^* - w^* \rangle \\ &\geq \|x - w\|^2 + \|x^* - w^*\|^2 - 2\|x - w\|\|x^* - w^*\| \geq 0, \end{aligned}$$

from which the rest of (10.1) now follows easily.

($\Longleftarrow$) Since (10.1) implies that $\langle x - w, x^* - w^* \rangle < 0$, this is immediate from the definition of maximal monotonicity. ▌

Remark 11. If E is not reflexive then there is no hope of getting a result analogous to Corollary 10. From James's theorem (see Pryce, [8] for a proof), there exists $w^* \in E^*$ that does not attain its norm on the unit ball of E. Let $M := E \times \{0\} \subset E \times E^*$. Then M is a maximal monotone subset of $E \times E^*$ and $(0, w^*) \in E \times E^* \setminus M$, but there does not exist $(x, x^*) \in M$ such that

$$x \neq 0,\ x^* \neq w^* \quad \text{and} \quad \langle x - 0, x^* - w^* \rangle = -\|x - 0\|\|x^* - w^*\|. \tag{10.1}$$

Since $(x, x^*) \in M \implies x^* = 0$, (10.1) would imply that $\langle x, w^* \rangle = \|x\|\|w^*\|$. Setting $b := x/\|x\|$, we would have $\|b\| = 1$ and $\langle b, w^* \rangle = \|w^*\|$, contradicting our choice of w^*. Now suppose that E is not reflexive, M is a maximal monotone subset of $E \times E^*$ and $(x, x^*) \in E \times E^* \setminus M$. It *may be* that there necessarily exists a sequence $\{(w_n, w_n^*)\}_{n \geq 1}$ of elements of M such that, for all $n \geq 1$, $w_n \neq x$, $w_n^* \neq x^*$ and

$$\frac{\langle w_n - x, w_n^* - x^* \rangle}{\|w_n - x\|\|w_n^* - x^*\|} \to -1 \quad \text{as } n \to \infty,$$

but we do not know if this is the case in general. It *is* the case if M is the graph of a continuous linear monotone operator, or the graph of the subdifferential of a proper, convex lower semicontinuous function.

The *duality map* $J\colon\ E \mapsto 2^{E^*}$ is defined by:

$$x^* \in Jx \iff \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2.$$

Furthermore, in accordance with the notation that we will introduce in the next section, we will write $G(-J) := \{(x, x^*)\colon\ x \in E,\ x^* \in -Jx\}$.

We leave to the reader the proof of the following lemma:

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Lemma 12. *Let $(x,x^*) \in E \times E^*$. Then*

$$(x,x^*) \in G(-J) \iff \|x\|^2 + \|x^*\|^2 + 2\langle x,x^*\rangle = 0.$$

Theorem 13 is the "$-J$" criterion for maximality.

Theorem 13. *Let E be reflexive and M be a monotone subset of $E \times E^*$. Then*

$$M \text{ is maximal monotone} \iff M + G(-J) = E \times E^*.$$

Proof. From Lemma 12 and Theorem 9, M is maximal monotone if, and only if, for all $(w,w^*) \in E \times E^*$, there exists $(x,x^*) \in M$ such that $(x-w,x^*-w^*) \in G(-J)$. But this last is equivalent to $(w-x,w^*-x^*) \in G(-J)$, i.e.,

$$(w,w^*) \in (x,x^*) + G(-J). \blacksquare$$

Applications to multifunctions

There is an obvious one–to–one correspondence between multifunctions from E into 2^{E^*} and subsets of $E \times E^*$: if $S\colon E \mapsto 2^{E^*}$ then $G(S)$ is the corresponding subset of $E \times E^*$, where

$$G(S) := \{(x,x^*)\colon\ x \in E,\ x^* \in Sx\}.$$

$G(S)$ is the "graph" of S. We shall always suppose that $G(S) \neq \emptyset$ — we shall emphasize this by saying that S is *nontrivial*. If $G \subset E \times E^*$, then the corresponding multifunction is defined by

$$Sx := \{x^* \in E^*\colon\ (x,x^*) \in G\}.$$

S is said to be *monotone* or *maximal monotone* if $G(S)$ has the corresponding property. We write

$$D(S) := \{x \in X\colon\ Sx \neq \emptyset\} \quad \text{and} \quad R(S) := \textstyle\bigcup_{x\in X} Sx.$$

$D(S)$ is the "domain" of S and $R(S)$ is the "range" of S. If $S_1\colon E \mapsto 2^{E^*}$ and $S_2\colon E \mapsto 2^{E^*}$ are nontrivial, we define $S_1 + S_2\colon E \mapsto 2^{E^*}$ (with $D(S_1+S_2) = D(S_1) \cap D(S_2)$) by

$$(S_1+S_2)x := S_1x + S_2x \quad (x \in E), \tag{13.1}$$

where $S_1x + S_2x$ is the "Minkowski sum" $\{x_1^* + x_2^*\colon\ x_1^* \in S_1x,\ x_2^* \in S_2x\}$. Now, it was proved by Minty that *if E is a Hilbert space and $S\colon E \mapsto 2^{E^*}$ is monotone then*

$$S \text{ is maximal monotone} \iff R(S+J) = E^*.$$

Rockafellar showed that Minty's result can be extended to the case where E is reflexive and J and J^{-1} are single–valued, where $J^{-1}\colon E^* \mapsto 2^E$ is defined by

$$J^{-1}x^* := \{x \in E\colon\ Jx \ni x^*\}.$$

Further, it was proved by Asplund that any reflexive Banach space can be renormed so that J and J^{-1} are single–valued. (Of course, renorming does not affect monotonicity or maximality). This renorming theorem is not easy. We show in Theorem 14 that the implication ($\Longrightarrow$), known as "Rockafellar's surjectivity theorem", is true even without the renorming.

Theorem 14. *Let E be reflexive and $S\colon\ E \mapsto 2^{E^*}$ be maximal monotone. Then*

$$R(S+J) = E^*.$$

Proof. Let $w^* \in E^*$. From Theorem 13,

$$(0, w^*) \in G(S) + G(-J).$$

Thus there exist $x \in E$, $x^* \in Sx$ and $y^* \in (-J)(-x)$ such that $x^* + y^* = w^*$. But then $y^* \in Jx$, hence

$$w^* = x^* + y^* \in Sx + Jx \subset R(S+J). \blacksquare$$

Remark 15. We outline the proof that if E is reflexive and J and J^{-1} are single–valued then

$$R(S+J) = E^* \quad \Longrightarrow \quad S \text{ is maximal monotone.} \tag{15.1}$$

Suppose that

$$(z, z^*) \in E \times E^* \quad \text{and} \quad \inf_{(s,s^*)\in G(S)} \langle s - z, s^* - z^* \rangle \geq 0.$$

Since $R(S+J) = E^*$, we can choose $(s, s^*) \in G(S)$ so that

$$s^* + Js = z^* + Jz. \tag{15.2}$$

(Remember that, for all $x \in E$, Jx is now a *singleton*.) We will show that

$$(z, z^*) = (s, s^*) \in G(S), \tag{15.3}$$

which will establish that S is maximal monotone. We have

$$\langle s - z, s^* - z^* \rangle + \langle s - z, Js - Jz \rangle = \langle s - z, (s^* + Js) - (z^* + Jz) \rangle = 0.$$

Since both terms on the left-hand side of the above equation are positive, they are both zero. In particular, $\langle s - z, Js - Jz \rangle = 0$, from which

$$\langle s, Js \rangle - \langle s, Jz \rangle - \langle z, Js \rangle + \langle z, Jz \rangle = 0, \tag{15.4}$$

which implies in turn that $\|s\|^2 - 2\|s\|\|z\| + \|z\|^2 \leq 0$. It follows from this that $\|z\| = \|s\|$, hence $\|Jz\| = \|Js\| = \|z\| = \|s\|$. Substituting this in (15.4),

$$2\|s\|\|z\| - \langle s, Jz \rangle - \langle z, Js \rangle = 0,$$

hence $\langle z, Js \rangle = \|z\|\|Js\| = \|z\|^2 = \|Js\|^2$, that is to say

$$Js = Jz. \tag{15.5}$$

Substituting this in (15.2), $s^* = z^*$. Since J^{-1} is single–valued, we also obtain from (15.5) that $s = z$. Thus we have established (15.3), as required.

It was pointed out by S. Fitzpatrick (personal communication) that *if J is not single–valued then (15.1) does not follow.* It was also pointed out by H. Bauschke (personal communication) that *if J is single–valued and J^{-1} is not single–valued then, again,* (15.1) *does not follow.* See [12] for details of their examples.

The maximal monotonicity of a sum — I

If $S_1\colon E \mapsto 2^{E^*}$ and $S_2\colon E \mapsto 2^{E^*}$ are nontrivial and monotone and $D(S_1) \cap D(S_2) \neq \emptyset$ then $S_1 + S_2$, defined in (13.1), is obviously nontrivial and monotone. On the other hand, it does *not* follow that if S_1 and S_2 are maximal monotone and $D(S_1) \cap D(S_2) \neq \emptyset$ then $S_1 + S_2$ is maximal monotone. (See Phelps, [7], p. 54.) Determining conditions on S_1 and S_2 (normally called "constraint qualifications") that ensure that $S_1 + S_2$ is maximal monotone is one of the fundamental questions in the theory of monotone multifunctions.

In order to gain insight into this problem, let us work backwards and suppose for the moment that $S_1 + S_2$ is maximal monotone. From Lemma 8(b), there exists $(z, z^*) \in G(S_1 + S_2)$ such that

$$\|z\|^2 + \|z^*\|^2 + 2\langle z, z^*\rangle = 0.$$

Let $(z, x_1^*) \in G(S_1)$, $(z, x_2^*) \in G(S_2)$ and $x_1^* + x_2^* = z^*$. Then

$$\|z\|^2 + \|x_1^* + x_2^*\|^2 + 2\langle z, x_1^* + x_2^*\rangle = 0$$

and, since S_1 and S_2 are monotone, for all $(s_1, s_1^*) \in G(S_1)$ and $(s_2, s_2^*) \in G(S_2)$,

$$\langle s_1 - z, s_1^* - x_1^*\rangle \geq 0 \quad \text{and} \quad \langle s_2 - z, s_2^* - x_2^*\rangle \geq 0.$$

Combining all this together, we see that

$$\begin{aligned}&(s_1, s_1^*) \in G(S_1) \text{ and } (s_2, s_2^*) \in G(S_2) \quad \Longrightarrow \\ &\qquad 2\langle s_1 - z, s_1^* - x_1^*\rangle + 2\langle s_2 - z, s_2^* - x_2^*\rangle \geq \|z\|^2 + \|x_1^* + x_2^*\|^2 + 2\langle z, x_1^* + x_2^*\rangle.\end{aligned}$$

The pivotal result in our investigation of the sum problem is Lemma 16, in which we use the fg–theorem, Theorem 3, to transform the above problem of finding one element of E and two elements of E^* into the problem of finding the one scalar constant γ. Though the important point in Lemma 16 is that (16.1) implies (16.2), the fact that it is an *equivalence* and the above discussion show that these conditions are not "pulled out of a hat". The problem is, of course, to find a value of γ satisfying (16.1). We will show in a later section how this can be achieved using Baire's theorem. For future reference, we shall call (16.1) the "γ–condition".

Lemma 16. *Let E be reflexive and $S_1\colon E \mapsto 2^{E^*}$ and $S_2\colon E \mapsto 2^{E^*}$ be nontrivial. Then the conditions* (16.1) *and* (16.2) *are equivalent.*

$$\left.\begin{aligned}&\textit{There exists } \gamma \geq 0 \textit{ such that} \quad (\mu_1, \mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2) \Longrightarrow \\ &\qquad 2r(\mu_1 + \mu_2) + 2\gamma\|p(\mu_2 - \mu_1)\| + \|p(\mu_1)\|^2 + \|q(\mu_1 + \mu_2)\|^2 \geq 0.\end{aligned}\right\} \tag{16.1}$$

$$\left.\begin{aligned} &\textit{There exist } x_1^*,\ x_2^* \in E^* \textit{ and } z \in E \textit{ such that}\\ &(s_1, s_1^*) \in G(S_1) \textit{ and } (s_2, s_2^*) \in G(S_2) \quad \Longrightarrow\\ &2\langle s_1 - z, s_1^* - x_1^*\rangle + 2\langle s_2 - z, s_2^* - x_2^*\rangle \ge\\ &\|z\|^2 + \|x_1^* + x_2^*\|^2 + 2\langle z, x_1^* + x_2^*\rangle. \end{aligned}\right\} \tag{16.2}$$

Proof. We shall establish the equivalence of (16.1) and (16.2) by proving their equivalence with the intermediate conditions (16.3) — (16.7) below:

$$\left.\begin{aligned} &\text{There exist } \gamma \ge 0 \text{ and } y^* \in (E \times E^*)^* \text{ such that}\\ &(\mu_1, \mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2) \quad \Longrightarrow\\ &2r(\mu_1 + \mu_2) + 2\gamma\|p(\mu_2 - \mu_1)\|\\ &\quad - 2\big\langle (p(\mu_1), q(\mu_1 + \mu_2)), y^*\big\rangle - \|y^*\|^2 \ge 0. \end{aligned}\right\} \tag{16.3}$$

$$\left.\begin{aligned} &\text{There exist } (z, z^*) \in E \times E^* \text{ and } \gamma \ge 0 \text{ such that}\\ &(\mu_1, \mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2) \quad \Longrightarrow\\ &2r(\mu_1 + \mu_2) + 2\gamma\|p(\mu_2 - \mu_1)\|\\ &\quad - 2\langle (p(\mu_1), z^*\rangle - 2\langle z, q(\mu_1 + \mu_2))\rangle - \|z\|^2 - \|z^*\|^2 \ge 0. \end{aligned}\right\} \tag{16.4}$$

$$\left.\begin{aligned} &\text{There exist } (z, z^*) \in E \times E^* \text{ and } x_2^* \in E^* \text{ such that}\\ &(\mu_1, \mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2) \quad \Longrightarrow\\ &2r(\mu_1 + \mu_2) - 2\langle p(\mu_2 - \mu_1), x_2^*\rangle\\ &\quad - 2\langle p(\mu_1), z^*\rangle - 2\langle z, q(\mu_1 + \mu_2)\rangle - \|z\|^2 - \|z^*\|^2 \ge 0. \end{aligned}\right\} \tag{16.5}$$

$$\left.\begin{aligned} &\text{There exist } x_1^*,\ x_2^* \in E^* \text{ and } z \in E \text{ such that}\\ &(\mu_1, \mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2) \quad \Longrightarrow\\ &2r(\mu_1 + \mu_2) - 2\langle p(\mu_1), x_1^*\rangle - 2\langle p(\mu_2), x_2^*\rangle\\ &\quad - 2\langle z, q(\mu_1 + \mu_2)\rangle - \|z\|^2 - \|x_1^* + x_2^*\|^2 \ge 0. \end{aligned}\right\} \tag{16.6}$$

$$\left.\begin{aligned} &\text{There exist } x_1^*,\ x_2^* \in E^* \text{ and } z \in E \text{ such that}\\ &(s_1, s_1^*) \in G(S_1) \text{ and } (s_2, s_2^*) \in G(S_2) \quad \Longrightarrow\\ &2\langle s_1, s_1^*\rangle + 2\langle s_2, s_2^*\rangle - 2\langle s_1, x_1^*\rangle - 2\langle s_2, x_2^*\rangle\\ &\quad - 2\langle z, s_1^* + s_2^*\rangle - \|z\|^2 - \|x_1^* + x_2^*\|^2 \ge 0. \end{aligned}\right\} \tag{16.7}$$

((16.1)$\Longleftrightarrow$(16.3)) This follows from the fg–theorem, Theorem 3, with $F := E \times E^*$, $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$, $A := \mathcal{CO}(S_1) \times \mathcal{CO}(S_2)$ and, for all $(\mu_1, \mu_2) \in A$,

$$f(\mu_1, \mu_2) := 2r(\mu_1 + \mu_2) + 2\gamma\|p(\mu_2 - \mu_1)\| \quad \text{and} \quad g(\mu_1, \mu_2) := (p(\mu_1), q(\mu_1 + \mu_2)).$$

((16.3)$\Longleftrightarrow$(16.4)) This is clear since any element y^* of F^* can be written in the form $(z^*, \widehat{z})$ for some $(z, z^*) \in E \times E^*$, and $\|y^*\| = \sqrt{\|z\|^2 + \|z^*\|^2}$.

((16.4)$\Longleftrightarrow$(16.5)) For fixed $(z,z^*) \in E \times E^*$, we define

$$h\colon\ \big(\mathcal{CO}(S_1) \times \mathcal{CO}(S_2)\big) \times E^* \mapsto \mathbb{R}$$

by

$$h((\mu_1,\mu_2),x^*) := 2r(\mu_1+\mu_2) - 2\langle p(\mu_2-\mu_1), x^*\rangle \\ - 2\langle p(\mu_1), z^*\rangle - 2\langle z, q(\mu_1+\mu_2)\rangle - \|z\|^2 - \|z^*\|^2.$$

Then the implication in (16.4) is simply that

$$\inf_A \max_B h \geq 0,$$

where $A := \mathcal{CO}(S_1) \times \mathcal{CO}(S_2)$, and $B := \{x^* \in E^*\colon\ \|x^*\| \leq \gamma\}$ with the topology $w(E^*,E)$. From the Banach–Alaoglu theorem, B is compact. The sets A and B are convex, h is affine on A, and affine and continuous on B. Thus, from the minimax theorem, Theorem 1, this is, in turn, equivalent to

$$\max_B \inf_A h \geq 0,$$

which is equivalent to (16.5).

((16.5)$\Longleftrightarrow$(16.6)) This follows from the substitution $x_1^* := z^* - x_2^*$.

((16.6)$\Longleftrightarrow$(16.7)) If x_1^*, x_2^* and z satisfy (16.6) then (16.7) follows by restricting μ_1 and μ_2 to the values $\delta_{(s_1,s_1^*)}$ and $\delta_{(s_2,s_2^*)}$. If, conversely, (16.7) is satisfied and $(\mu_1,\mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2)$ then (16.6) follows by multiplying the left side of the inequality in (16.7) by $\mu_1(s_1,s_1^*)\mu_2(s_2,s_2^*)$ and summing up over all $(s_1,s_1^*) \in G(S_1)$ and $(s_2,s_2^*) \in G(S_2)$.

((16.7)$\Longleftrightarrow$(16.2)) This can be seen by rearranging the terms and adding $\pm 2\langle z, x_1^* + x_2^*\rangle$ to both sides.

This completes the proof of Lemma 16. ▌

We now start considering the converse problem, which is actually what interests us. That is to say, we assume the γ–condition, (16.1), and move towards the proof that S_1+S_2 is maximal monotone. Our first step in this direction will be Lemma 17, which is derived from a technique due to Minty and Browder (see [2], Lemma 6, p. 99). We will then continue this analysis in the last section of the paper.

Lemma 17. *Let E be reflexive and $S_1\colon\ E \mapsto 2^{E^*}$ and $S_2\colon\ E \mapsto 2^{E^*}$ be maximal monotone and satisfy the γ–condition, (16.1). Then there exists $(z,z^*) \in G(S_1+S_2)$ such that*

$$\|z\|^2 + \|z^*\|^2 + 2\langle z, z^*\rangle = 0. \tag{17.1}$$

Proof. It follows from Lemma 16, that there exist $x_1^*,\ x_2^* \in E^*$ and $z \in E$ so that, for all $(s_1,s_1^*) \in G(S_1)$ and $(s_2,s_2^*) \in G(S_2)$,

$$2\langle s_1 - z, s_1^* - x_1^*\rangle + 2\langle s_2 - z, s_2^* - x_2^*\rangle \geq \|z\|^2 + \|x_1^* + x_2^*\|^2 + 2\langle z, x_1^* + x_2^*\rangle.$$

Putting $z^* := x_1^* + x_2^*$, this can be rewritten

$$2\langle s_1 - z, s_1^* - x_1^*\rangle + 2\langle s_2 - z, s_2^* - x_2^*\rangle \ge \|z\|^2 + \|z^*\|^2 + 2\langle z, z^*\rangle.$$

Taking the infimum over all $(s_1, s_1^*) \in G(S_1)$ and $(s_2, s_2^*) \in G(S_2)$ and using the perfect square trick, Lemma 2,

$$\begin{aligned} 2 \inf_{(s,s^*)\in G(S_1)} \langle s - z, s^* - x_1^*\rangle + 2 \inf_{(s,s^*)\in G(S_2)} \langle s - z, s^* - x_2^*\rangle \\ \ge \|z\|^2 + \|z^*\|^2 + 2\langle z, z^*\rangle \ge 0. \end{aligned}$$

From Lemma 6(c), both these infima are zero hence, from Lemma 6(b), $(z, x_1^*) \in G(S_1)$ and $(z, x_2^*) \in G(S_2)$. Consequently,

$$(z, z^*) = (z, x_1^* + x_2^*) \in G(S_1 + S_2).$$

This completes the proof of Lemma 17. ▮

The dom lemma

We will use Baire's theorem in order to find the constant γ for Lemma 17. The most convenient way to approach this is through the properties of convex lower semicontinuous functions. In this section, we give the "dom lemma", Lemma 19, which is one such property, and which we shall use in Lemma 23.

Let $A \subset E$. A is said to be *absorbing* if $\bigcup_{\lambda>0} \lambda A = E$. We start off with a classical result.

Lemma 18. *Let C be a closed convex absorbing set in E. Then C is a neighborhood of* 0.

Proof. Let $D := C \cap -C$. Then D is closed, convex and absorbing and $D = -D$, i.e., D is a "barrel". The result follows by applying Kelly–Namioka, [5], p. 104 to D.

Lemma 19. *Let f: $E \mapsto \mathbb{R} \cup \{\infty\}$ be a convex, lower semicontinuous function, and let* $\operatorname{dom} f$ *be absorbing. Then there exist $\eta > 0$ and $n \ge 1$ such that*

$$w \in E \text{ and } \|w\| \le \eta \quad \Longrightarrow \quad f(w) \le n. \tag{19.1}$$

Proof. Since $\operatorname{dom} f$ is absorbing, $0 \in \operatorname{dom} f$. Let $n \ge f(0) \vee 0 + 1$, and put

$$C := \{w \in E\colon\ f(w) \le n\}.$$

C is clearly closed and convex. We now show that C is absorbing. To this end, let w be an arbitrary element of E. Since $\operatorname{dom} f$ is absorbing, there exist $\lambda > 0$ and $x \in \operatorname{dom} f$ such that $\lambda w = x$. We choose $\mu \in (0, 1]$ so that

$$\mu(f(x) - n + 1) \le 1.$$

Then

$$\begin{aligned} f(\mu\lambda w) &= f(\mu x) \\ &\le \mu f(x) + (1-\mu) f(0) \\ &\le \mu f(x) + (1-\mu)(n-1) \\ &= \mu(f(x) - n + 1) + n - 1 \le n. \end{aligned}$$

In other words, $\mu\lambda w \in C$. Thus C is absorbing, as required. It now follows from Lemma 18 that C is a neighborhood of 0, which gives the required result. ▮

Remark 20. The classical "uniform boundedness theorem" can easily be deduced from the dom lemma. So also can the result that a convex lower semicontinuous function is locally bounded on the interior of its domain. (See, for instance, Phelps, [7], Proposition 3.3, p. 39.)

The third technique: the function χ_S

We suppose throughout this section that $S\colon E \mapsto 2^{E^*}$ is a nontrivial multifunction. We shall use $\mathcal{CO}(S)$ as an abbreviation for $\mathcal{CO}(G(S))$.

Definition 21. We define the function $\chi_S\colon E \mapsto \mathbb{R} \cup \{\infty\}$ by

$$\chi_S(x) := \sup_{\mu \in \mathcal{CO}(S)} \frac{\langle x, q(\mu)\rangle - r(\mu)}{1 + \|p(\mu)\|} \quad (x \in E).$$

Since χ_S is the supremum of a family of continuous affine functions, it is convex and lower semicontinuous. The function χ_S was first defined, and many of its properties were established in the paper [3] by Coodey–Simons. The "D–dom lemma", Lemma 22 shows that if S is monotone then $\operatorname{dom}\chi_S \neq \emptyset$. We use the notation "co" to stand for "convex hull of".

Lemma 22. *Let $S\colon E \mapsto 2^{E^*}$ be nontrivial and monotone. Then*

$$D(S) \subset \operatorname{co} D(S) \subset \operatorname{dom}\chi_S.$$

Proof. Since $\operatorname{dom}\chi_S$ is convex, it remains to prove that

$$D(S) \subset \operatorname{dom}\chi_S. \tag{22.1}$$

To this end, let $w \in D(S)$. Pick $w^* \in Sw$, and define $\beta := \langle w, w^*\rangle \vee \|w^*\|$. If $(s, s^*) \in G(S)$ then, since S is monotone,

$$\langle w, s^*\rangle - \langle s, s^*\rangle = \langle w - s, s^*\rangle \le \langle w - s, w^*\rangle = \langle w, w^*\rangle - \langle s, w^*\rangle.$$

Thus, for all $\mu \in \mathcal{CO}(S)$ and $(s, s^*) \in G(S)$,

$$\langle w, \mu_{(s,s^*)}s^*\rangle - \mu_{(s,s^*)}\langle s, s^*\rangle \le \mu_{(s,s^*)}\langle w, w^*\rangle - \langle \mu_{(s,s^*)}s, w^*\rangle.$$

Summing over $(s, s^*) \in G(S)$,

$$\begin{aligned}\langle w, q(\mu)\rangle - r(\mu) &\le \langle w, w^*\rangle - \langle p(\mu), w^*\rangle \\ &\le \langle w, w^*\rangle + \|p(\mu)\|\|w^*\| \\ &\le \beta(1 + \|p(\mu)\|).\end{aligned}$$

Dividing by $1 + \|p(\mu)\|$, and taking the supremum over $\mu \in \mathcal{CO}(S)$, we see that $\chi_S(w) \le \beta$, which implies that $w \in \operatorname{dom}\chi_S$. This completes the proof of (22.1), and hence that of Lemma 22. ▌

A generalization of the local boundedness theorem

Lemma 23 is an equivalence valid for any nontrivial multifunction on a Banach space. We will use it in Lemma 24 as part of our proof of Rockafellar's sum theorem. It also extends the following result proved by Borwein–Fitzpatrick in [1]: *Let S be nontrivial and monotone, and v be an "absorbing point" of $D(S)$. Then S is locally bounded at v.*

Lemma 23. *Let $S\colon E \mapsto 2^{E^*}$ be nontrivial and $v \in E$. Then $\operatorname{dom}\chi_S - v$ is absorbing $\iff$ there exist $\eta > 0$ and $n \geq 1$ such that*

$$\mu \in \mathcal{CO}(S) \quad \Longrightarrow \quad r(\mu) + n(1 + \|p(\mu)\|) \geq \langle v, q(\mu)\rangle + \eta\|q(\mu)\|. \tag{23.1}$$

Proof. ($\Longrightarrow$) From the dom lemma, Lemma 19, there exist $\eta > 0$ and $n \geq 1$ such that

$$w \in E \text{ and } \|w\| \leq \eta \quad \Longrightarrow \quad \chi_S(v + w) \leq n.$$

Thus, from the definition of χ_S,

$$\left.\begin{aligned} &w \in E, \quad \|w\| \leq \eta \quad \text{and} \quad \mu \in \mathcal{CO}(S) \quad \Longrightarrow \\ &\qquad\qquad \langle v + w, q(\mu)\rangle - r(\mu) \leq n(1 + \|p(\mu)\|), \end{aligned}\right\}$$

that is to say,

$$\left.\begin{aligned} &\mu \in \mathcal{CO}(S), \quad w \in E \quad \text{and} \quad \|w\| \leq \eta \quad \Longrightarrow \\ &\qquad\qquad r(\mu) + n(1 + \|p(\mu)\|) \geq \langle v, q(\mu)\rangle + \langle w, q(\mu)\rangle. \end{aligned}\right\}$$

We now obtain (23.1) by taking the supremum of the right-hand expression over all $w \in E$ such that $\|w\| \leq \eta$.

($\Longleftarrow$) If η and n satisfy (23.1) then, reversing the above steps,

$$w \in E \text{ and } \|w\| \leq \eta \quad \Longrightarrow \quad \chi_S(v + w) \leq n \quad \Longrightarrow \quad v + w \in \operatorname{dom}\chi_S. \blacksquare$$

The maximal monotonicity of a sum — II

Rockafellar's original result (see [9], Theorem 1, p. 76) was that if E is reflexive, S_1 and S_2 are maximal monotone and

$$D(S_1) \cap \operatorname{int} D(S_2) \neq \emptyset \tag{23.2}$$

then $S_1 + S_2$ is maximal monotone. In fact, we shall prove in Theorem 26 that (23.2) can be weakened to

$$\operatorname{co} D(S_1) \cap \operatorname{int}\operatorname{co} D(S_2) \neq \emptyset.$$

Our first step towards this, Lemma 24, assumes neither the reflexivity of E nor the maximal monotonicity of S_1 or S_2.

Lemma 24. *Let $S_1\colon E \mapsto 2^{E^*}$ and $S_2\colon E \mapsto 2^{E^*}$ be nontrivial and monotone, and $\operatorname{co} D(S_1) \cap \operatorname{int}\operatorname{co} D(S_2) \neq \emptyset$. Then:*

$$\left.\begin{aligned} &\textit{there exists } \gamma \geq 0 \textit{ such that} \quad (\mu_1, \mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2) \quad \Longrightarrow \\ &\quad 2r(\mu_1 + \mu_2) + 2\gamma\|p(\mu_2 - \mu_1)\| + \|p(\mu_1)\|^2 + \|q(\mu_1 + \mu_2)\|^2 \geq 0. \end{aligned}\right\} \tag{16.1}$$

Proof. To simplify the expressions in what follows, we write χ_i instead of χ_{S_i}. Let $v \in \operatorname{co} D(S_1) \cap \operatorname{int} \operatorname{co} D(S_2)$. From Lemma 22, $v \in \operatorname{dom} \chi_1 \cap \operatorname{int} \operatorname{dom} \chi_2$. From Lemma 23, there exist $n \geq 1$ and $\eta \in (0,1]$ such that

$$\mu_2 \in \mathcal{CO}(S_2) \quad \Longrightarrow \quad r(\mu_2) + n(1 + \|p(\mu_2)\|) \geq \langle v, q(\mu_2)\rangle + \eta\|q(\mu_2)\|. \tag{24.1}$$

By increasing n if necessary, we can also suppose that

$$n \geq \chi_1(v) \quad \text{and} \quad n \geq \|v\|.$$

We write $\gamma := 5n^2/\eta$, and we shall show that γ has the required property. So let $(\mu_1, \mu_2) \in \mathcal{CO}(S_1) \times \mathcal{CO}(S_2)$. If $\|q(\mu_2)\| \leq \gamma$, then (16.1) follows from Corollary 5(b), so we only have to consider the other alternative, namely that $\|q(\mu_2)\| > \gamma$. We derive then from (24.1) that

$$r(\mu_2) + n(1 + \|p(\mu_2)\|) - \langle v, q(\mu_2)\rangle - 5n^2 \geq 0. \tag{24.2}$$

Since $\chi_1(v) \leq n$,

$$r(\mu_1) + n(1 + \|p(\mu_1)\|) - \langle v, q(\mu_1)\rangle \geq 0. \tag{24.3}$$

Adding (24.2) and (24.3) and using the fact that $\|p(\mu_2)\| \leq \|p(\mu_1)\| + \|p(\mu_2 - \mu_1)\|$,

$$r(\mu_1 + \mu_2) + n\|p(\mu_2 - \mu_1)\| + 2n\|p(\mu_1)\| - \langle v, q(\mu_1 + \mu_2)\rangle - 3n^2 \geq 0.$$

Multiplying by 2 and using the fact that $\|v\| \leq n$,

$$2r(\mu_1 + \mu_2) + 2n\|p(\mu_2 - \mu_1)\| + 4n\|p(\mu_1)\| + 2n\|q(\mu_1 + \mu_2)\| - 6n^2 \geq 0.$$

Since $4n\|p(\mu_1)\| \leq 4n^2 + \|p(\mu_1)\|^2$ and $2n\|q(\mu_1 + \mu_2)\| \leq n^2 + \|q(\mu_1 + \mu_2)\|^2$,

$$2r(\mu_1 + \mu_2) + 2n\|p(\mu_2 - \mu_1)\| + \|p(\mu_1)\|^2 + \|q(\mu_1 + \mu_2)\|^2 - n^2 \geq 0.$$

(16.1) follows from this since $n \leq \gamma$. ∎

In the next lemma, we combine the results of Lemmas 17 and 24, and also carry out some simple bootstrapping steps.

Lemma 25. *Let E be reflexive.*
(a) Let $S_1\colon E \mapsto 2^{E^*}$ and $S_2\colon E \mapsto 2^{E^*}$ be maximal monotone and

$$\operatorname{co} D(S_1) \cap \operatorname{int} \operatorname{co} D(S_2) \neq \emptyset. \tag{25.1}$$

Then there exists $(z, z^*) \in G(S_1 + S_2)$ such that

$$\|z\|^2 + \|z^*\|^2 + 2\langle z, z^*\rangle = 0.$$

(b) Let $T_1\colon E \mapsto 2^{E^*}$ and $T_2\colon E \mapsto 2^{E^*}$ be maximal monotone and

$$\operatorname{co} D(T_1) \cap \operatorname{int} \operatorname{co} D(T_2) \neq \emptyset. \tag{25.2}$$

Suppose also that $w \in E$. *Then there exists* $(\zeta, z^*) \in G(T_1 + T_2)$ *such that*

$$\|\zeta - w\|^2 + \|z^*\|^2 + 2\langle \zeta - w, z^* \rangle = 0.$$

(c) Let $S_1\colon E \mapsto 2^{E^*}$ *and* $S_2\colon E \mapsto 2^{E^*}$ *be maximal monotone and*

$$\operatorname{co} D(S_1) \cap \operatorname{int} \operatorname{co} D(S_2) \neq \emptyset.$$

Suppose also that $w \in E$ *and* $w^* \in E^*$. *Then there exists* $(\zeta, \zeta^*) \in G(S_1 + S_2)$ *such that*

$$\|\zeta - w\|^2 + \|\zeta^* - w^*\|^2 + 2\langle \zeta - w, \zeta^* - w^* \rangle = 0.$$

Proof. (a) is immediate from Lemmas 24 and 17.
(b) Let $S_i := (T_i^{-1} - w)^{-1}$. Then S_1 and S_2 satisfy (25.1). The result follows from part (a), with $\zeta := z + w$.
(c) Let $T_i := S_i - w^*/2$. Then T_1 and T_2 satisfy (25.2). The result follows from part (b), with $\zeta^* := z^* + w^*$. ▌

Theorem 26. *Let E be reflexive,* $S_1\colon E \mapsto 2^{E^*}$ *and* $S_2\colon E \mapsto 2^{E^*}$ *be maximal monotone and*

$$\operatorname{co} D(S_1) \cap \operatorname{int} \operatorname{co} D(S_2) \neq \emptyset.$$

Then $S_1 + S_2$ *is maximal monotone.*

Proof. This is immediate from Lemma 25(c) and Theorem 9. ▌

We note from Lemma 12 that the conclusion of Lemma 25(a) is that $R(S_1 + S_2 + J) \ni 0$. If $w \in E$, define $J_w(x) := J(x - w) \quad (x \in E)$. Then the conclusion of Lemma 25(c) is that,

$$\text{for all } w \in E, \quad R(S_1 + S_2 + J_w) = E^*.$$

The simple bootstrapping procedure outlined in Lemma 25(b–c) enables us to avoid having to use a renorming theorem. (To be more precise, what we avoid is the use of the result that *E can be renormed so that if* $S\colon E \mapsto 2^{E^*}$ *is monotone and* $S + J$ *is surjective then* S *is maximal monotone*. See the comments preceding Theorem 14.) We also point out for the record that we do not use any fixed–point theorems either.

Remark 27. An examination of the proof we have given shows that we can, in fact, prove the following: *Let E be reflexive,* $S_1\colon E \mapsto 2^{E^*}$ *and* $S_2\colon E \mapsto 2^{E^*}$ *be maximal monotone and* $\operatorname{dom} \chi_{S_1} \cap \operatorname{int} \operatorname{dom} \chi_{S_2} \neq \emptyset$. *Then* $S_1 + S_2$ *is maximal monotone.* However, the bootstrapping carried out in Lemma 25 becomes a little more technical. The following much harder results are proved in [12]: *Let E be reflexive,* $S_1\colon E \mapsto 2^{E^*}$ *and* $S_2\colon E \mapsto 2^{E^*}$ *be maximal monotone and*

$$\bigcup_{\lambda > 0} \lambda\big[\operatorname{co} D(S_1) - \operatorname{co} D(S_2)\big] \quad \textit{be a closed subspace}$$

or

$$\bigcup_{\lambda>0} \lambda\big[\operatorname{dom}\chi_{S_1} - \operatorname{dom}\chi_{S_2}\big] \quad \textit{be a closed subspace.}$$

Then S_1+S_2 is maximal monotone. These results rely on the following extension of Lemma 19: *Let g_1, g_2 be convex and lower semicontinuous and* $\operatorname{dom} g_1 - \operatorname{dom} g_2$ *be absorbing. Then there exists $n \geq 1$ such that*

$$\{x \in E\colon\ g_1(x) \leq n\} - \{x \in E\colon\ g_2(x) \leq n\} \quad \textit{is a neighborhood of } 0.$$

There is at least one significant case where the constant γ for Lemma 17 can be found *without* the use of Baire's theorem. This is in the analysis of the *Brézis–Crandall–Pazy condition* for the maximal monotonicity of a sum. See [12] for details of this approach.

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