MINIMAX THEOREMS

We suppose that X and Y are nonempty sets and $f\colon\ X \times Y \mapsto \mathbb{R}$. A *minimax theorem* is a theorem that asserts that, under certain conditions,

$$\inf_Y \sup_X f = \sup_X \inf_Y f,$$

that is to say,

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \inf_{y\in Y} f(x,y).$$

The purpose of this article is to give the reader the flavor of the different kind of minimax theorems, and of the techniques that have been used to prove them. This is a very large area, and it would be impossible to touch on all the work that has been done in it in the space that we have at our disposal. The choice that we have made is to give the historical roots of the subject, and then go directly to the most recent results. The reader who is interested in a more complete narrative can refer to the 1974 survey article [35] by E.B.Yanovskaya, the 1981 survey article [8] by I.Irle and the 1995 survey article [28] by S.Simons.

von Neumann's results.

In his investigation of *games of strategy* , J.von Neumann realized that, even though a two–person zero–sum game did not necessarily have a solution in *pure* strategies, it did have to have one in *mixed* strategies. Here is a statement of that seminal result ([32], translated into English in [33]):

Theorem 1 (1928). *Let A be an $m \times n$ matrix, and X and Y be the sets of nonnegative row and column vectors with unit sum. Then*

$$\min_{y\in Y} \max_{x\in X} xAy = \max_{x\in X} \min_{y\in Y} xAy.$$

Despite the fact that the statement of this result is quite elementary, the proof was quite sophisticated, and depended on an extremely ingenious induction argument. Nine years later in [34], von Neumann showed that the bilinear character of Theorem 1 was not needed when he extended it as follows, using *Brouwer's fixed–point theorem*:

Theorem 2 (1937). *Let X and Y be nonempty compact, convex subsets of Euclidean spaces, and $f\colon\ X \times Y \mapsto \mathbb{R}$ be jointly continuous. Suppose that f is quasiconcave on X and quasiconvex on Y (see below). Then*

$$\min_Y \max_X f = \max_X \min_Y f.$$

When we say that f is *quasiconcave* on X, we mean that

$$\text{for all } y \in Y \text{ and } \lambda \in \mathbb{R},\ GT(\lambda, y) \text{ is convex,}$$

and when we say that$f$ is *quasiconvex* on Y, we mean that

$$\text{for all } x \in X \text{ and } \lambda \in \mathbb{R},\ LE(x, \lambda) \text{ is convex.}$$

Here, $GT(\lambda, y)$ and $LE(x, \lambda)$ are "level sets" associated with the function f. Specifically,

$$GT(\lambda, y) := \big\{x \in X\colon\ f(x,y) > \lambda\big\}$$

and

$$LE(x, \lambda) := \big\{y \in Y\colon\ f(x,y) \le \lambda\big\}.$$

In 1941, S.Kakutani [10] analyzed von Neumann's proof and, as a result, discovered the fixed-point theorem that bears his name.

Infinite–dimensional results for convex sets.

minimax theorem
E.B.Yanovskaya
I.Irle
S.Simons
→ *game of strategy*
J.von Neumann
Brouwer's fixed–point theorem
→ *compact set*
→ *convex set*
→ *quasiconcave function*
→ *quasiconvex function*
S.Kakutani

The first infinite–dimensional minimax theorem was proved in 1952 by K.Fan ([1]), who generalized Theorem 2 to the case when X and Y are compact, convex subsets of infinite dimensional locally convex spaces, and the quasiconcave and quasiconvex conditions are somewhat relaxed. The result in this general line that has the simplest statement is that of M.Sion, who proved the following ([30]):

Theorem 3 (1958). *Let X be a convex subset of a linear topological space, Y be a compact convex subset of a linear topological space, and $f\colon\ X \times Y \mapsto \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that f is quasiconcave on X and quasiconvex on Y. Then*

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

When we say that f is "upper semicontinuous on X" and "lower semicontinuous on Y" we mean that, for all $y \in Y$, the map $x \mapsto f(x, y)$ is *upper semicontinuous* and, for all $x \in X$, the map $y \mapsto f(x, y)$ is *lower semicontinuous*. The importance of Sion's weakening of continuity to semicontinuity was that it indicated that many kind of minimax problems had equivalent formulations in terms of subsets of X and Y, and led to Fan's 1972 work ([4]) on sets with convex sections and minimax inequalities, which has since found many applications in economic theory. Like Theorem 2, all these result relied ultimately on Brouwer's fixed–point theorem (or the related *Knaster–Kuratowski–Mazurkiewicz (KKM) lemma* on closed subsets of a finite dimensional simplex).

Functional–analytic minimax theorems.

The first person to take minimax theorems out of the context of convex subsets of vector spaces, and their proofs (other than that of the matrix case discussed in Theorem 1) out of the context of fixed–point theorems was K.Fan in 1953 ([2]). We present here a generalization of Fan's result due to H.König ([16]). König's proof depended on the Mazur–Orlicz version of the Hahn–Banach theorem (see below).

Theorem 4 (1968). *Let X be a nonempty set and Y be a nonempty compact topological space. Let $f\colon\ X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose that:*

for all $x_1, x_2 \in X$, there exists $x_3 \in X$ such that

$$f(x_3, .) \geq \frac{f(x_1, .) + f(x_2, .)}{2} \text{ on } Y,$$

and,

for all $y_1, y_2 \in Y$, there exists $y_3 \in Y$ such that

$$f(., y_3) \leq \frac{f(., y_1) + f(., y_2)}{2} \text{ on } X.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

We give here the statement of the *Mazur–Orlicz version of the Hahn–Banach theorem*, since it is a very useful result and it not as well–known as it deserves to be. *Let S be a sublinear functional on a real vector space E, and C be a nonempty convex subset of E. Then there exists a linear functional L on E such that $L \leq S$ on E and $\inf_C L = \inf_C S$.* See [17], [19] and [20] for applications of the Mazur–Orlicz theorem and the related "sandwich theorem" to measure theory, Hardy algebra theory and the theory of flows in infinite networks.

The kind of minimax theorem discussed in this section (where X is not topologized) has turned out to be extremely useful in functional analysis, in particular in *convex analysis* and also in the theory of *monotone operators on a*

K.Fan
M.Sion
linear topological space
→ *upper semicontinuous function*
→ *lower semicontinuous function*
Knaster–Kuratowski–Mazurkiewicz (KKM) lemma
K.Fan
H.König
→ *Mazur–Orlicz theorem*
convex analysis

Banach space. (See [29] for more details of these kinds of applications.)

Minimax theorems that depend on connectedness.

It was believed for some time that proofs of minimax theorems required either the fixed–point machinery of algebraic topology, or the functional–analytic machinery of convexity. However, in 1959, W.-T.Wu proved the first minimax theorem in which the conditions of convexity were totally replaced by conditions related to *connectedness*. This line of research was continued by H.Tuy, L.L.Stachó, M.A.Geraghty with B.-L.Lin, and J.Kindler with R.Trost, whose results were all subsumed by a family of general topological minimax theorem established by H.König in [18]. Here is a typical result from [18]. In order to simplify the statements of this and some of our later results, we shall write $f_* := \sup_X \inf_Y f$. f_* is the "lower value" of f. If $\lambda \in \mathbb{R}$, $V \subset Y$ and $W \subset X$, we write $GT(\lambda, V) := \bigcap_{y \in V} GT(\lambda, y)$ and $LE(W, \lambda) := \bigcap_{x \in W} LE(x, \lambda)$.

Theorem 5 (1992). *Let X be a connected topological space, Y be a compact topological space, and $f\colon\ X \times Y \mapsto \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Let Λ be a nonempty subset of (f_*, ∞) such that $\inf \Lambda = f_*$ and suppose that, for all $\lambda \in \Lambda$, for all nonempty subsets V of Y, and for all nonempty finite subsets W of X,*

$$GT(\lambda, V) \text{ is connected in } X,$$

and

$$LE(W, \lambda) \text{ is connected in } Y.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

Mixed minimax theorems.

In [31], F.Terkelsen proved the first *mixed* minimax theorem. We describe Terkelsen's result as "mixed" since one of the conditions in it is taken from Theorem 4, and the other from Theorem 5:

Theorem 6 (1972). *Let X be a nonempty set and Y be a nonempty compact topological space. Let $f\colon\ X \times Y \to \mathbb{R}$ be lower semicontinuous on Y. Suppose that,*

for all $x_1, x_2 \in X$, there exists $x_3 \in X$ such that

$$f(x_3, .) \geq \frac{f(x_1, .) + f(x_2, .)}{2} \text{ on } Y.$$

Suppose also that, for all $\lambda \in \mathbb{R}$ and, for all nonempty finite subsets W of X,

$$LE(W, \lambda) \text{ is connected in } Y.$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

A metaminimax theorem.

It was believed for some time that Brouwer's fixed-point theorem or the Knaster-Kuratowski-Mazurkiewicz lemma was required to order to prove Sion's Theorem, Theorem 3. However, In 1966, M.A.Ghouila-Houri ([7]) proved Theorem 3 using a simple combinatorial property of convex sets in finite dimensional space. This

→ *monotone operator on a Banach space*
W.-T.Wu
connectedness
H.Tuy
L.L.Stachó
M.A.Geraghty
B.-L.Lin
J.Kindler
R.Trost
H.König
F.Terkelsen
M.A.Ghouila-Houri

was probably the first indication of the breakdown of the classification of minimax theorems as either of “topological” or “functional–analytic” type. Further indication of this breakdown was provided by Terkelsen’s result, Theorem 6, and the subsequent 1982 results of I.Joó and L.L.Stachó ([9]), the 1985 and 1986 results of M.A.Geraghty and B.-L.Lin ([5] and [6]), and the 1989 results of H.Komiya ([15]).

J.Kindler ([11]) was the first to realize (in 1990) that some abstract concept akin to connectedness might be involved in minimax theorems, even when the topological condition of connectedness was not explicitly assumed. This idea was pursued by S.Simons with the introduction in 1992 of the concept of *pseudoconnectedness*, which we will now describe. We say that sets H_0 and H_1 are *joined* by a set H if

$$H \subset H_0 \cup H_1, \quad H \cap H_0 \neq \emptyset$$
and
$$H \cap H_1 \neq \emptyset.$$

We say that a family $\mathcal{H}$ of sets is *pseudoconnected* if:

$$H_0, H_1, H \in \mathcal{H} \text{ and } H_0 \text{ and } H_1 \text{ joined by } H$$
$$\Longrightarrow$$
$$H_0 \cap H_1 \neq \emptyset.$$

Any family of closed connected subsets of a topological space is pseudoconnected. So also is any family of open connected subsets. However, pseudoconnectedness can be defined in the absence of any topological structure and, as we shall see in Theorem 7, is closely related to minimax theorems. Theorem 7 is the improvement of the result of [26] due to H.König — see [27]. We shall say that a subset W of X is *good* if W is finite and,

$$\text{for all } x \in X,\ LE(x, f_*) \cap LE(W, f_*) \neq \emptyset.$$

I.Joó
L.L.Stachó
M.A.Geraghty
B.-L.Lin
H.Komiya
J.Kindler
S.Simons
H.König
J.Kindler

Theorem 7 (1995). *Let Y be a topological space, and Λ be a nonempty subset of $\mathbb{R}$ such that* $\inf \Lambda = f_*$. *Suppose that, for all $\lambda \in \Lambda$ and for all good subsets W of X,*

$$\textit{for all } x \in X, LE(x, \lambda) \textit{ is closed and compact,}$$
$$\{LE(x, \lambda) \cap LE(W, \lambda)\}_{x \in X} \textit{ is pseudoconnected}$$

and, for all $x_0, x_1 \in X$, there exists $x \in X$ such that

$$LE(x_0, \lambda) \textit{ and } LE(x_1, \lambda) \textit{ are joined by}$$
$$LE(x, \lambda) \cap LE(W, \lambda).$$

Then

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

Theorem 7 is proved by induction on the cardinality of the good subsets of W. Given the obvious topological motivation behind the concept of pseudoconnectedness, it is hardly surprising that Theorem 7 implies Theorem 5. What is more unexpected is that Theorem 7 implies Theorems 4 and 6 also. We prefer to describe Theorem 7 as a *metaminimax theorem* rather than a *minimax theorem*, since it is frequently harder to prove that the conditions of Theorem 7 are satisfied in any particular case that it is to prove Theorem 7 itself. So Theorem 7 is really a device for obtaining minimax theorems rather than a minimax theorem in its own right.

More recent work by J.Kindler ([12], [13] and [14]) on abstract intersection theorems has been at the interface between minimax theory and abstract set theory.

Minimax theorems and weak compactness.

There are close connections between minimax theorems and *weak compactness*. The following "converse minimax theorem" was proved by S.Simons in [22] — this result also shows that there are limitations on the extent to which one can totally remove the assumption of compactness from minimax theorems.

Theorem 8 (1971). *Suppose that X is a nonempty bounded, convex, complete subset of a locally convex space E with dual space E^*, and*

$$\inf_{y\in Y}\sup_{x\in X}\langle x,y\rangle = \sup_{x\in X}\inf_{y\in Y}\langle x,y\rangle$$

whenever Y is a nonempty convex, equicontinuous, subset of E^*. Then

X is weakly compact.

No compactness is assumed in the following, much harder, result (see [23]):

Theorem 9 (1972). *If X is a nonempty bounded, convex subset of a locally convex space E such that every element of the dual space E^* attains its supremum on X, and Y is any nonempty convex equicontinuous subset of E^*, then*

$$\inf_{y\in Y}\sup_{x\in X}\langle x,y\rangle = \sup_{x\in X}\inf_{y\in Y}\langle x,y\rangle.$$

If one now combines the results of Theorems 8 and 9, one can obtain a proof of the "sup theorem" of R.C.James, one of the most beautiful results in functional analysis: *if C is a nonempty bounded closed convex subset of E then C is $w(E,E^*)$–compact if, and only if, for all $x^*\in E^*$, there exists $x\in C$ such that* $\langle x,x^*\rangle = \max_C x^*$. James's theorem is not easy — the standard proof can be found in the paper [21] by J.D.Pryce.

We refer the reader to [28] for more details of the connections between minimax theorems and weak compactness.

Minimax inequalities for two or more functions.

Motivated by *Nash equilibrium* and the theory of *non–cooperative games*, K.Fan generalized Theorem 2 to the case of more than one function. In particular, he proved in [3] the following two-function minimax inequality (since the compactness of X is not needed, this result can in fact be strengthened to include Sion's theorem, Theorem 3, by taking $g=f$):

Theorem 10 (1964). *Let X and Y be nonempty compact, convex subsets of topological vector spaces and $f,g: X\times Y\mapsto\mathbb{R}$. Suppose that f is lower semicontinuous on Y and quasiconcave on X, g is upper semicontinuous on X and quasiconvex on Y, and*

$$f\le g \text{ on } X\times Y.$$

Then

$$\min_Y\sup_X f\le\sup_X\inf_Y g.$$

K.Fan (unpublished) and S.Simons (see [24]) generalized König's theorem, Theorem 4, with the following two–function minimax inequality:

Theorem 11 (1981). *Let X be a nonempty set, Y be a compact topological space and $f,g: X\times Y\mapsto\mathbb{R}$. Suppose that f is lower semicontinuous on Y,*

for all $y_1,y_2\in Y$, there exists $y_3\in Y$ such that

$$f(.,y_3)\le\frac{f(.,y_1)+f(.,y_2)}{2} \text{ on } X,$$

for all $x_1,x_2\in X$, there exists $x_3\in X$ such that

$$g(x_3,.)\ge\frac{g(x_1,.)+g(x_2,.)}{2} \text{ on } Y,$$

weak compactness
S.Simons
R.C.James
J.D.Pryce
Nash equilibrium
→ *non–cooperative game*
K.Fan
K.Fan
S.Simons

and

$$f \le g \text{ on } X \times Y.$$

Then

$$\min_Y \sup_X f \le \sup_X \inf_Y g.$$

Theorems 10 and 11 both unify the theory of minimax theorems and the theory of *variational inequalities*. The curious feature about these two results is that they have *opposite geometric pictures*. This question is discussed in [24] and [25]. The relationship between Theorem 10 and Brouwer's fixed-point theorem is quite interesting. As we have already pointed out, Sion's theorem, Theorem 3, can be proved in an elementary fashion without recourse to fixed-point related concepts. On the other hand, Theorem 10 can, in fact, be used to prove *Tychonoff's fixed–point theorem*, which is itself a generalization of Brouwer's fixed–point theorem. (See [3] for more details of this.)

A number of authors have proved minimax inequalities for more than two functions. We refer the reader to [28] for more details of these results.

Coincidence theorems.

A *coincidence theorem* is a theorem that asserts that if $S\colon\ X \mapsto 2^Y$ and $T\colon\ Y \mapsto 2^X$ have nonempty values and satisfy certain other conditions then there exist $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in Sx_0$ and $x_0 \in Ty_0$. The connection with minimax theorems is as follows: suppose that $\inf_Y \sup_X f \ne \sup_X \inf_Y f$. Then there exists $\lambda \in \mathbb{R}$ such that

$$\sup_X \inf_Y f < \lambda < \inf_Y \sup_X f.$$

Hence,

$$\text{for all } x \in X, \text{ there exists } y \in Y \text{ such that } f(x,y) < \lambda$$

and,

$$\text{for all } y \in Y, \text{ there exists } x \in X \text{ such that } f(x,y) > \lambda.$$

→ *variational inequality*
Tychonoff's fixed–point theorem
coincidence theorem

Define $S\colon\ X \mapsto 2^Y$ and $T\colon\ Y \mapsto 2^X$ by

$$Sx := \{y \in Y\colon\ f(x,y) < \lambda\} \ne \emptyset$$

and

$$Tx := \{x \in X\colon\ f(x,y) > \lambda\} \ne \emptyset.$$

If S and T were to satisfy a coincidence theorem then we would have $x_0 \in X$ and $y_0 \in Y$ such that

$$f(x_0,y_0) < \lambda \quad \text{and} \quad f(x_0,y_0) > \lambda,$$

which is clearly impossible. Thus this coincidence theorem would imply that

$$\inf_Y \sup_X f = \sup_X \inf_Y f.$$

The coincidence theorems known in algebraic topology consequently give rise to corresponding minimax theorems. There is a very extensive literature about coincidence theorems. We refer the reader to [28] for more details about this.

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