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Math 5B Midterm, July 16th, 2012

Instructions: Read the instructions for each question carefully. Answers without work or explanation will receive no credit. Feel free to ask questions if part of the test needs clarification.

1. Fill in the blanks in the following sentences (6 points).

   (a) For a function $f : \mathbb{R}^3 \to \mathbb{R}$, the gradient of f at a point (a, b, c) is a 3 dimensional vector, which is orthogonal to a level set of $f(x, y, z)$.

   (b) For a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, the derivative γ' at a point $a \in \mathbb{R}$ is geometrically the tangent vector to γ at $\gamma(a)$.

   (c) $\vec{v} \cdot \vec{u} =$ $\|\vec{v}\|\|\vec{u}\| \cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{u}$.

2. Mark whether the following are true or false (you can just write T for true or F for false). If the answer is false, explain what you could change to make it true (6 points).

(a) F For any differentiable function $f(x, y)$, we know that $f_{xy} = f_{yx}$.

$f(x,y)$ must be C^2 to guarantee that $f_{xy} = f_{yx}$.

(b) F If $A = \{(x, y) \mid x \geq 1\}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on A, then we know that f attains an absolute max / min on A

A is not bounded.

(c) T If $A = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on A, then we know that f attains an absolute max / min on A

3. (a) Find the equation of the tangent plane to the surface $f(x, y) = x^3y + 2x - 2y$ at the point $(1, 1, 1)$. Leave your answer in the form $z = ax + by + c$ (7 points).

$$\nabla f(x,y) = (3x^2y + 2,\ x^3 - 2)$$

$$\nabla f(1,1) = (5,\ -1)$$

Equation for tangent plane is:

$$z = f(1,1) + \nabla f(1,1)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

$$= 1 + 5(x-1) - (y-1)$$

$$= 5x - y - 3$$

$$\boxed{z = 5x - y - 3}$$

(b) Find the equation of a plane parallel to the one in part (a), but which passes through the origin (3 points).

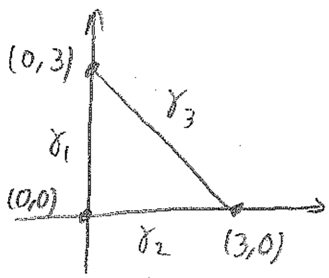
Two planes are parallel if they have the same (or co-linear) normal vector.

$$\boxed{z = 5x - y}$$

Can check this contains the origin by plugging in $x = y = 0$.

4. Find the absolute max and min of the function $f(x, y) = (x-1)^2 + (y-1)^2 + 3$ on the closed region bounded by the triangle with vertices $(0,0)$, $(0,3)$, and $(3,0)$ (10 points).

Be sure to show your work and/or give explanations!



1) Check interior of region.

$f_x(x,y) = 2x - 2$

$f_y(x,y) = 2y - 2$

$\rightsquigarrow$ critical point at $(1,1)$

$f(1,1) = 3$

2) Check sides

$\gamma_1 = (0, t)$ $\quad f(t, 0) = (t-1)^2 + 4$

$\frac{d}{dt} f(t,0) = 2(t-1)$

$\rightsquigarrow t=1$ crit $\rightsquigarrow (0,1)$ critical point, $f(0,1) = 4$

$\gamma_2 = (t, 0)$ $\quad f(t,0) = (t-1)^2 + 4$

$\rightsquigarrow t = 1$ crit $\rightsquigarrow (1,0)$ critical point, $f(1,0) = 4,$

$\gamma_3 = (t, 3-t)$ $\quad f(t, 3-t) = (t-1)^2 + (2-t)^2 + 3$

$\frac{d}{dt} f(t, 3-t) = 2(t-1) - 2(2-t) = 4t - 6$

$\rightsquigarrow t = 3/2$ crit $\rightsquigarrow (3/2, 3/2)$ critical point, $f(3/2, 3/2) = 7/2$

3) Check vertices

$f(0,0) = 5, \quad f(0,3) = 8, \quad f(3,0) = 8$

$\boxed{\text{Absolute max: } 8 \quad \text{Absolute min: } 3}$

5. Consider the function $f(x, y) = xy + e^{-xy}$ (10 points total).

(a) Find a vector orthogonal to the ~~level surface~~ graph of f ~~which contains~~ at the point $(1, -1, -1+e)$

$$g(x,y,z) = xy + e^{-xy} - z$$

$$\nabla g = (y - ye^{-xy},\ x - xe^{-xy},\ -1)$$

$$\boxed{\nabla g(1,-1,-1+e) = (-1+e,\ 1-e,\ -1)}$$

(b) At the point $(1, 2, 2+e^{-2})$, in what direction is $f(x, y)$ increasing most rapidly? In what direction is it decreasing most rapidly?

$$\nabla f = (y - ye^{-xy},\ x - xe^{-xy})$$

Greatest increase is $\nabla f(1,2) = (2 - 2e^{-2},\ 1 - e^{-2})$

Greatest decrease is $-\nabla f(1,2) = (-2 + 2e^{-2},\ -1 + e^{-2})$

(c) At the point $(1, 2, 2+e^{-2})$, find a vector $\vec{u}$ so that the rate of change of $f(x, y)$ is equal to 0 in the direction of this vector.

Since $D_u f = \nabla f \cdot \hat{u}$, we need to find $\vec{u} = (a, b) \neq \vec{0}$ so that $\nabla f(1,2) \cdot \vec{u} = 0$. Since we can scale $\vec{u}$ by any constant, assume $a = 1$ (here we're guessing that $a \neq 0$).

Then: $\nabla f(1,2) \cdot (1, b) = 2 - 2e^{-2} + b(1 - e^{-2})$

$$\Rightarrow b = \frac{2e^{-2} - 2}{1 - e^{-2}}$$

$$\boxed{\text{So } u = \left(1, \frac{2e^{-2}-2}{1-e^{-2}}\right)}$$

(d) Let $f(x,y) = (2xy + x^2, e^x + e^y)$. Find the derivative Df (5 points).

$$Df = \begin{bmatrix} 2y + 2x & 2x \\ e^x & e^y \end{bmatrix}$$

BONUS: What does it mean for a multivariable function to be differentiable? An intuitive answer is fine, I'm not expecting for you to have memorized the formula.

It has a "good" linear approximation at every point.

If $f: \mathbb{R}^2 \to \mathbb{R}$, this is the same as saying that the graph of f has a unique tangent plane at every point.