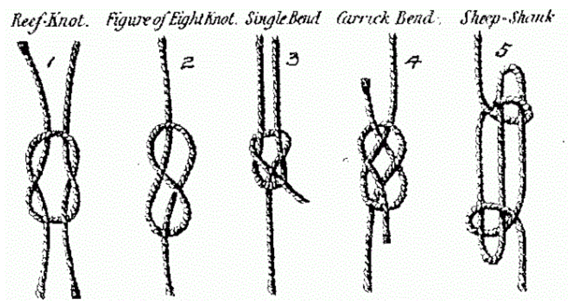
Math 7H Professor: Padraic Bartlett

# Lecture 1: Knot Theory

*Week 1* *UCSB 2015*

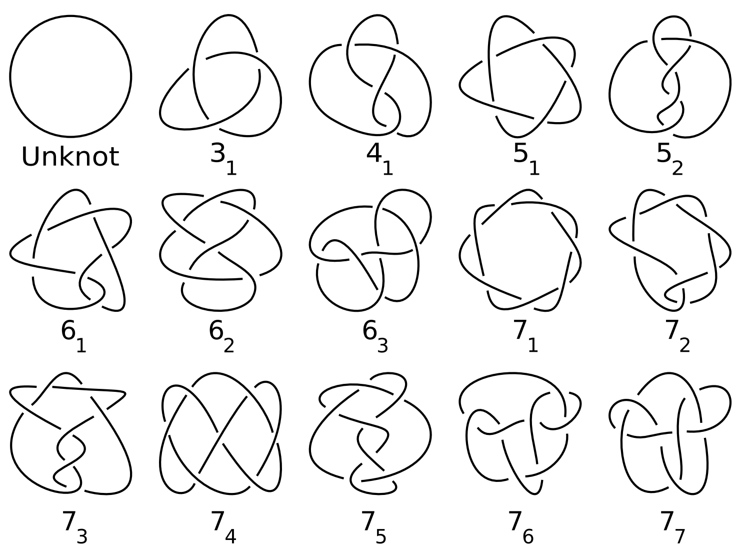
## 1 Introduction

Outside of mathematics, **knots** are ways to loop a single piece of string around itself:




In mathematics, we mean something slightly different by a knot. Instead of "a way to tie a rope," we will define a **mathematical knot** to be any way to embed[1] a circle in $\mathbb{R}^3$. Intuitively, think of an "embedding" as any way of smoothly drawing a curve in $\mathbb{R}^3$, so that you end back at where you started.

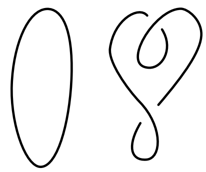
Even though knots are three-dimensional things, we can draw them on paper:




[1] An **embedding** of a circle in $\mathbb{R}^3$ is any map $\varphi : [0,1] \to \mathbb{R}^3$ that is differentiable (that is, the derivative of $\varphi$ exists everywhere) and injective (that is, the graph of $\varphi$ doesn't intersect itself,) such that $\varphi(0) = \varphi(1)$ (in other words, we start where we end, which is why we have a closed loop.) We will not usually need to be this formal with working with knots, but if you're curious, this is how we formally define things!

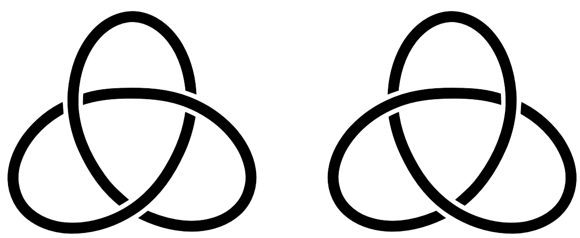
Notice how we convey all of the information about what parts of the knot go "above" or "below" other parts by creating a break in the strand that goes under.

We consider two knots to be **equivalent** if there is some way to stretch or move[2] one of the knots around in space, without tearing or passing parts of the knot through itself, so that we get the second knot. Again, this is a concept best thought of intuitively: imagine both of the loops below are made out of rubber bands.



You can clearly move and stretch the left loop around in space to get the right one; consequently, we think of these two loops as equivalent.

It would seem like many knots are different: for example, the "unknot," which is just the circle, should be different from the trefoil (labeled $3_1$ above.) But how can we prove this? How can we distinguish other kinds of knots?



The "left" and "right" trefoils. Can you deform space in such a way that one of these knots is transformed into the other?

In this class, we will discuss one particularly beautiful way to distinguish some classes of knots: the concept of tricolorability!

## 2 Reidemeister Moves

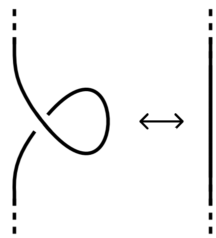
To do this, we need to introduce some rigorous notions of what it means to "move" a knot around. We do this via the Reidemeister moves:

**Definition.** Take any knot drawn in space. There are three special kinds of ways to manipulate our knot:

[2]Formally speaking, by "stretch and move" we mean any deformation of $\mathbb{R}^3$. By *deformation*, we mean the following: suppose that $\varphi_1 : [0,1] \to \mathbb{R}^3, \varphi_2 : [0,1] \to \mathbb{R}^3$ are two knots, as formally defined earlier. A **deformation** of the first knot into the second is any differentiable and injective map $\Phi : [0,1] \times [0,1] \to \mathbb{R}^3$ such that $\Phi(x,0) = \varphi_1(x)$ and $\Phi(x,1) = \varphi_2(x)$. Basically, think of $\Phi(x,t)$ as a map whose second argument $t$ is "time:" when you plug in $t = 0$, you get the first knot, when you plug in $t = 1$ you get the second knot, and as you let $t$ go from 0 to 1 you can watch the first knot move/stretch into the second knot! Again, this is something where the informal idea of "move one knot until it is the other knot" is going to be easier for us to work with instead of this more formal definition.

1. If we have one strand in our knot, we can twist or untwist that strand.
2. If we have two strands of our knot, we can push one strand over the other or not.
3. If we have a crossing of two strings somewhere in our knot, and some third thread goes over the top of that crossing, we can move that thread to either side of our crossing.

These moves, as with most things in knot theory, are best explained via pictures:



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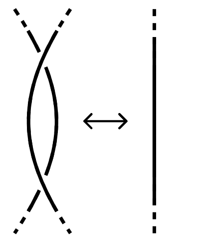

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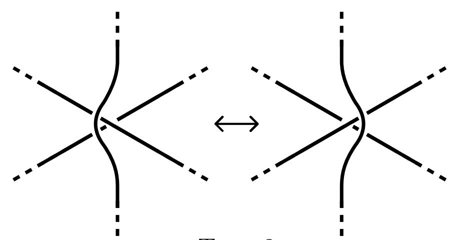


Type 3

Reidemeister moves are interesting mostly because of the following two theorems:

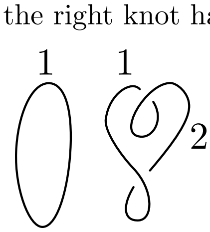
**Theorem.** If we take any knot and perform a series of Reidemeister moves on that knot, then the resulting knot is equivalent to the original knot. (This should make sense: all of the moves above can be done by just moving and stretching a knot in space!)

**Theorem.** If two knots are equivalent, then we can turn either one into the other via a sequence of Reidemeister moves. (This is the surprising theorem: any way of moving / repositioning knots in space can be accomplished with just some list of the above moves!)

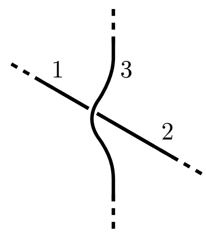
This is not too difficult to prove, but it is somewhat beyond our techniques right now. Instead of proving this result, I'll focus on what we can do with it: **tricolorability!**

## 3 Tricolorability

**Definition.** Take a knot, and draw it in $\mathbb{R}^2$. Call this drawing $T$. The **strands** of $T$ are all of the segments of the knot consisting of unbroken lines. For example, the left knot below consists of exactly one strand, while the right knot has two strands:



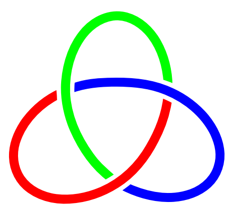
Notice that at any crossing in our knot drawing, there are three strands that meet at that crossing (with some strands possibly repeated): the one strand that goes over, and the two strands that correspond to the "under."



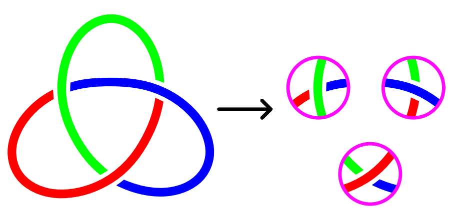

**Definition.** Given a knot drawn in the plane, call its drawing $T$. We say that this drawing is **tricolorable** if there is some way to assign each strand of $T$ one of the three colors $R, G, B$ so that

- All three colors get used somewhere in our knot diagram.
- At any crossing, either all three of the crossing strands are the same color, or all three are different colors.

**Example.** The trefoil knot is a tricolorable knot. To see why, simply look at the following picture:



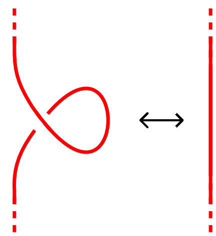
In the picture above, we used all three colors to color this knot; moreover, at each crossing, we have that all three crossing strands are different colors.



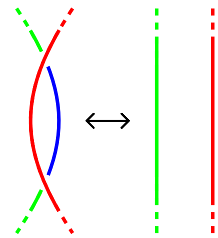
The reason tricolorability is interesting is that, even though it was defined for specific ways to draw a knot, it is a knot **invariant**: in other words, if a knot is tricolorable, you can tricolor **any** drawing of that knot! This, surprisingly, is not a hard thing to prove:

**Theorem.** Suppose that $T_1, T_2$ are two drawings of equivalent knots. Then $T_1$ is tricolorable if and only if $T_2$ is tricolorable.

*Proof.* If $T_1, T_2$ are two drawings of equivalent knots, then there is some sequence of Reidemeister moves that turn $T_1$ into $T_2$, and vice-versa. Now, notice the following: if we have a tricolored knot, and we perform any of the Reidemeister moves, we still have a tricolored knot! We show this is true via the following diagram:



Type 1



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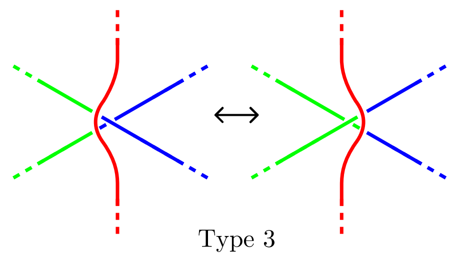

Type 3

Given any tricolored knot, the diagram above shows how performing Reidemeister moves does not change the fact that our knot is tricolored; in all three diagrams, our knot starts and ends as a tricolored knot, without changing the colors of the ends of our strands. This tells us that the tricoloring we have on all of the parts of our knot that we didn't touch has been preserved, and therefore that Reidemeister moves preserve tricolorability!

Therefore, if $T_1$ was a tricolorable knot, we could turn $T_1$ into a tricoloring of $T_2$, by just performing a sequence of Reidemeister moves to turn $T_1$ into $T_2$; similarly, we can turn tricolorings of $T_2$ into $T_1$. In any case, we have proven our claim: $T_1$ is tricolorable if and only if $T_2$ is tricolorable! $\square$

Using this theorem, we can answer a question we started class with:

**Theorem.** The unknot is not equivalent to the trefoil.

*Proof.* The trefoil is tricolorable; however, the unknot is not, as it has only one strand (and thus cannot have a tricoloring, as tricolorings need three colors!) $\square$