Math 7H	Professor: Padraic Bartlett

Homework 10: Generating Functions and Dice

Due Tuesday of finals week, by 5pm at my office — *UCSB 2015*

Hey! This HW is an extra-credit assignment if you wanted points or things to do. Check it out and have fun!

To work with the dice we studied in class, we needed to factor certain polynomials into irreducible factors. Here are some helpful deus-ex-machina claims that you can use without proof on the next few problems:

Observation.

$$x^n - 1 = \prod_{\substack{\text{all } d \text{ such that} \\ d \geq 1 \text{ and } d \text{ divides } n}} \Phi_d(x),$$

where the polynomials $\Phi_d(x)$ are the cyclotomic[1] polynomials

Observation. The cyclotomic polynomials are all irreducible as polynomials over the integers. In other words, we cannot factor any cyclotomic polynomial into two smaller-degree integer polynomials!

Observation. The following is a list of some of the cyclotomic polynomials:

$$\begin{aligned}
\Phi_1(x) &= x - 1 & \Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 & \Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2x + x + 1 & \Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 & \Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^3 + 1 & \Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\
\Phi_{12}(x) &= x^4 - x^2 + 1 & \Phi_{20}(x) &= x^8 - x^6 + x^4 - x^2 + 1
\end{aligned}$$

1. Using the list of polynomials above, can you classify all of the pairs of nonstandard k-dice that are "the same" as a pair of standard k-dice when rolled and summed?

   (To make our lives easier, assume that we are working with typical physical dice here, so that our values of k are limited to the platonic solids. In other words, we only need to consider $k = 4, 6, 8, 12$, or 20, as those are the number of faces on a tetrahedron/cube/octahedron/dodecahedron/icosahedron.)

[1] The d-th cyclotomic polynomial is given by the formula

$$\Phi_d(x) = \prod_{\omega}(x - \omega),$$

where the product above is taken over all primitive d-th roots of unity. A primitive d-th root of unity is a complex number ω such that $\omega^d = 1$ and $\omega^i \neq 1$, for any $1 \leq i < d$.

For this problem, you don't need to know what these things are; I've given you a list of all of the cyclotomic polynomials you need! But it's still cool to run into.