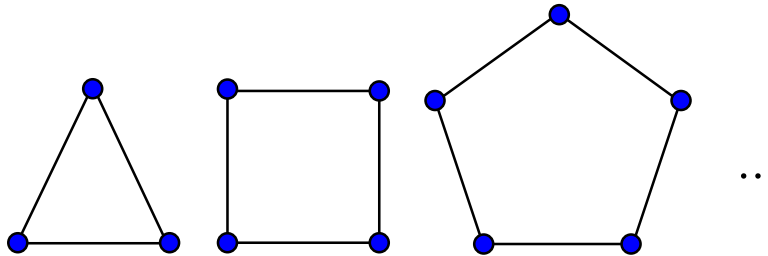


## Lecture 9: Trees and Art Galleries

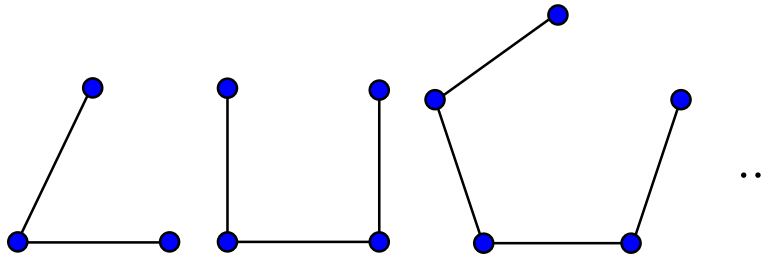
## 1 Prelude: Graph Theory

This talk uses the mathematical concepts of **graphs** from our previous class. In particular, it refers to the concepts of **trees** and **cycles**, which we redefine here:

**Definition.** The **cycle graph** on  $n$  vertices,  $C_n$ , is the graph on the vertex set  $\{v_1, v_2, \dots, v_n\}$  with edge set  $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$ . The cycle graphs  $C_n$  can be drawn as  $n$ -gons, as depicted below:



**Definition.** The **path graph** on  $n$  vertices,  $P_n$ , is the graph on the vertex set  $\{v_1, v_2, \dots, v_n\}$  with edge set  $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}\}$ . The path graphs  $P_n$  can be drawn as paths of length  $n$ , as depicted below:

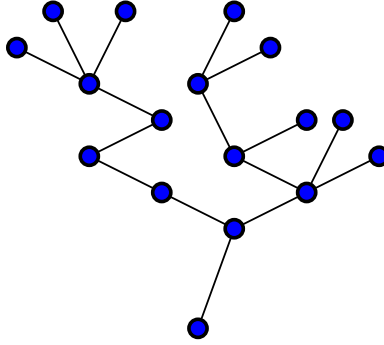


**Definition.** Given a graph  $G$  and another graph  $H$ , we say that  $H$  is a **subgraph** of  $G$  if and only if  $V(H) \subset V(G)$  and  $E(H) \subset E(G)$ .

**Definition.** Given a graph  $G$ , we call  $G$  **connected** if for any two vertices  $x, y \in V(G)$ , there is a path that starts at  $x$  and ends at  $y$  in our graph  $G$ .

**Definition.** If a graph  $G$  has no subgraphs that are cycle graphs, we call  $G$  **acyclic**. A **tree**  $T$  is a graph that's both connected and acyclic. In a tree, a **leaf** is a vertex whose degree is 1.

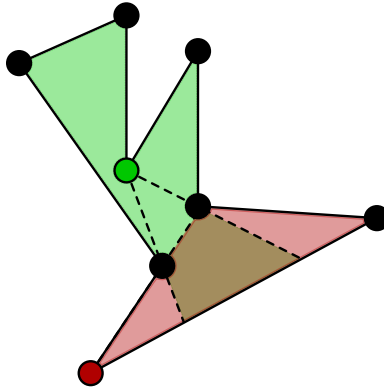
**Example.** The following graph is a tree:



With these terms restated, we can move to today's problem: the Art Gallery Problem!

## 2 The Art Gallery Problem

**Question 1.** *Suppose that you have an art gallery that is shaped like some sort of  $n$ -polygon, and you want to place cameras with  $360^\circ$ -viewing angles along the vertices of your polygon in such a way that the entire gallery is under surveillance. How many cameras do you need?*



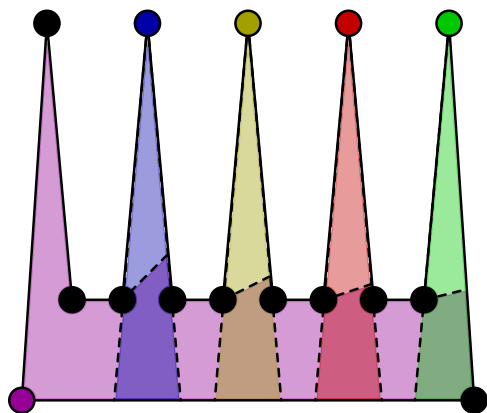
A gallery guarded by 2 guards, Red and Green.

One trivial upper bound you can come up with is  $n$  guards: just put one guard on each vertex of our polygon with  $n$  sides!

Can we do better? As it turns out, we can!

**Claim.** (Chvátal) You need at most  $\lfloor n/3 \rfloor$ -many cameras to guard a  $n$ -polygon.

It bears noting that this bound of  $\lfloor n/3 \rfloor$  is sharp. Consider the following art gallery:



A crown-shaped art gallery.

In this sort-of “crown-shaped” art gallery, each prong of the crown (i.e. triangle) needs to have a guard on one of its three vertices to guard the entire triangle, as no other vertices can “see” the entirety of that prong. Therefore, you need one guard for each prong; i.e.  $n/3$  guards, for a crown with  $n/3$  prongs (i.e.  $n$  vertices.)

To prove Chvátal’s theorem, we need a few lemmas first:

**Lemma 2.** *If  $G$  is a  $n$ -polygon with  $n \geq 4$ , then there is some line segment formed by two of the vertices in  $G$  that lies entirely in  $G$ .*

*Proof.* Let  $v$  be the leftmost vertex of  $G$ . (If there is a tie, take  $v$  to be the top leftmost vertex of  $G$ .) Let  $u$  and  $w$  be  $v$ ’s neighbors, and examine the line segment  $\overline{uw}$ . If this lies entirely in  $G$ , great! Otherwise, it must cross some edge of  $G$ ; consequently, there must be a vertex of  $G$  that lies inside of the triangle spanned by the three points  $u, v, w$ . Let  $x$  be the vertex furthest from the line segment  $\overline{uw}$  that lies in this triangle. Then, look at the line segment  $\overline{vx}$ ; because  $x$  is the furthest point in  $\Delta uvw$  from  $\overline{uw}$ , there can’t be any edges of  $G$  that are crossed by this line segment (as one of their endpoints would necessarily be closer to  $v$ .) So  $\overline{vx}$  lies entirely in  $G$ .  $\square$

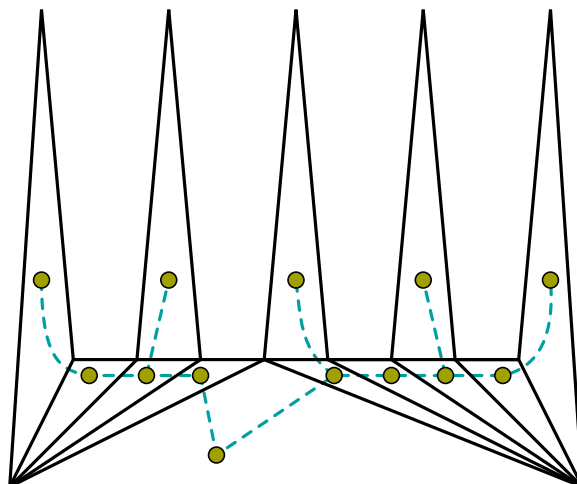
**Corollary 3.** *Any  $n$ -polygon can be divided into  $n - 2$ -triangles.*

*Proof.* Using the process above, repeatedly divide our  $n$ -polygon into a pair of smaller polygons, one with  $k$  vertices and the other with  $n + 2 - k$  vertices, until all of these polygons are triangles. By induction, it is not hard to see that the number of these triangles is  $n - 2$ .  $\square$

So: we can turn any polygon into a number of triangles stuck to each other! We use this to turn any art gallery on  $n$  vertices into a **graph** on  $n - 2$  vertices, as follows:

- Start by taking our polygon  $G$  and turning it into a collection  $\{T_i\}_{i=1}^{n-2}$  of triangles.
- For each triangle  $T_i$ , associate a vertex  $t_i$ .
- Connect two vertices  $t_i, t_j$  with an edge if their corresponding triangles  $T_i, T_j$  share a face.

Call this graph  $T'$  the **dual graph** of  $T$ .



Turning the crown into a tree.

This is a graph! Furthermore, it's a fairly special kind of graph: it's a tree! We prove this here:

**Lemma 4.** *Let  $G$  be a polygon,  $T$  be a triangulation of  $G$  performed as above, and let  $T'$  be the dual graph to this triangulation (i.e. put a vertex in the center of every face of  $T$ , and connect two faces iff they share an edge.) This graph is a tree.*

*Proof.* Let  $T$  be our triangulated polygon. In our construction above, each of the edges of  $T'$  corresponds to a diagonal of the polygon  $G$ , that divides our polygon into two distinct smaller polygons. Because cutting our polygon  $G$  along one of those diagonals will always divide the polygon into two disconnected pieces, doing so will always result in two triangles that are no longer connected by a chain of triangles with adjacent faces!

In other words: in the dual graph  $T'$  that we made above, removing any edge disconnects our graph! So our graph has the following properties:

- There is a path between any two vertices because our polygon is connected.
- There is at most one path between any two vertices, because if there were two paths and we cut an edge on one path, our graph would not be disconnected (we could simply use the other path to connect vertices!) In other words, our graph is **acyclic**, because a cycle exists in our graph if and only if there are two different paths connecting two vertices! (Prove this to yourself if you are skeptical.)

Therefore  $T$  is connected and acyclic: i.e. it's a tree! □

This tree is remarkably useful; in particular, we can use its structure to create a system for assigning guards! We do this here:

**Lemma 5.** *Take a polygon  $G$  that has been triangulated as described earlier. Then we can color each of the vertices of  $G$  either red, blue or green, so that each triangle contains one vertex of each color.*

*Proof.* For our triangulated polygon  $G$ , take the dual graph/tree  $T'$  that we constructed above, and pick some vertex  $t_0$  in it. For all relevant integers  $k$ , let  $T'_k$  be the collection of vertices that are distance  $k$  away from  $v$ , for every  $k$ . (The **distance** of two vertices from each other is the length of the shortest path between them.)

Color the vertices of  $T$  as follows:

- Take the triangle in  $G$  associated to  $t_0$  and color its three vertices red, green and blue.
- Suppose we've colored all of the vertices attached to triangles with corresponding vertices in  $T'_i$ , for some  $i$ . Now, look at the triangles corresponding to vertices in  $T'_{i+1}$ . Each triangle associated to a vertex  $t_{i+1}$  in this set shares exactly one edge with some triangle associated to a vertex in  $T'_i$ ; this is because if our vertex is distance  $i + 1$  from  $t_0$ , then (by taking the path of distance  $i + 1$  and walking one step closer to  $t_0$ ) there is an adjacent vertex (and thus face-sharing triangle) at distance  $i$ , i.e. in  $T'_i$ .

Furthermore, because  $T'$  is a tree, there is exactly one edge from any  $t_{i+1}$  to vertices in the set  $\bigcup_{j=0}^{i+1} T'_j$ . This is because the existence of any other edge would create a cycle, because it would give us two distinct paths to  $t_0$ !

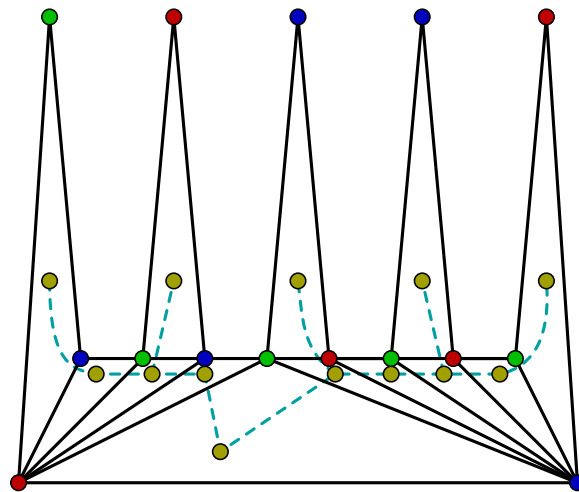
Therefore, the triangle associated to  $t_{i+1}$  shares a face with **only one other** triangle in all of the sets that we've already colored! Therefore, only two of its vertices have been assigned colors. Thus, there is always some spare third color to use to color its third vertex! Use this to color its third vertex, and repeat for all vertices in  $T_{i+1}$ .

- Repeat until every vertex in  $T$  is colored. Note that each triangle has one vertex of each color.

□

**Corollary 6.** *You need at most  $\lfloor n/3 \rfloor$ -many cameras to guard a  $n$ -polygon.*

*Proof.* By the above, create a triangulation and 3-coloring of our polygon  $G$  with the colors  $\{R, G, B\}$ . Now, station guards at whichever color is used the least number of times in this triangulation! Each guard can see everything in their assigned triangle by construction. Therefore, the entire art gallery is guarded. □



To guard this crown, simply pick one of (red, green, blue,) and station guards at vertices of that color.