Math 7H	Professor: Padraic Bartlett
Lecture 7: Graph Theory and the Four-Color Theorem	
Week 7	*UCSB 2015*

In this class, we are going to prove the four-color theorem! It goes as follows:

Theorem. Take any map, which for our purposes is a way to partition the plane $\mathbb{R}^2$ into a collection of connected regions $R_1, \ldots R_n$ with continuous boundaries. There is some way to assign each region R_i to a color in the set $\{R, G, B, Y\}$, such that if two regions R_i, R_j are "touching" (i.e. they share some nonzero length of boundary between them,) then those two regions must receive different colors.

This might not seem like a graph theory question as phrased above! We fix this here, through the following notion of a "dual" graph to a map:

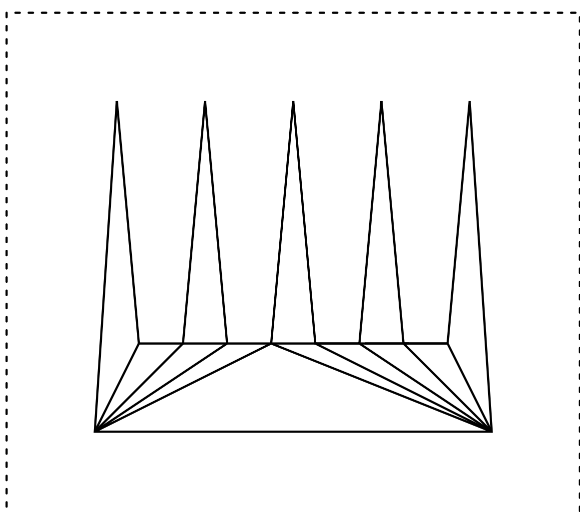
Definition. Take any map M. We can turn this into a graph as follows:

- Assign to each region R_i a vertex v_i.
- Connect v_i to v_j with an edge if the regions R_i, R_j are touching.

We call this graph the **dual graph** to M.

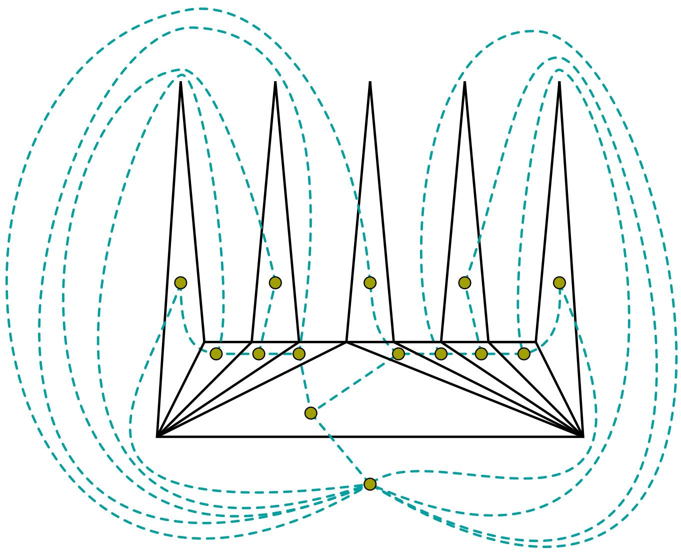
We give an example here:

Example. Consider the following map:



This map consists of 14 regions. If you count, you can see that the figure drawn consists thirteen triangles; as well, we have the "outer" region consisting of everything else left over, which forms a very strange 15-gon.

Now, take each region, and assign to it a vertex. As well, connect two regions sharing a border with an edge: this will give you the following graph, with edges given by the dashed teal lines and vertices given by the yellow dots:



Notice that whenever we do this, the resulting graph is **planar** (think about why this is true!) So, with this observation, we can "turn" our four-color theorem from above into the following graph-theoretic claim:

Theorem. If G is a connected planar graph on finitely many vertices, then $\chi(G) \leq 4$.

We prove this here! In particular, we present Kempe's proof of the four-color theorem:

1 Kempe's Proof

We start with a quick lemma:

Lemma. Take any connected planar graph G. Then there is some vertex v in our graph with degree at most 5.

Proof. We proceed by contradiction. Assume that every vertex has at least degree 6. Last week, we proved that for any connected planar graph G, $V - E + F = 2$. We create a contradiction involving this property here.

First, consider the sum $\sum_{v \in G} \deg(v)$. On one hand, this is twice the number of edges in G: this is because each edge shows up twice in this sum (once for each endpoint v when we're calculating $\deg(v)$.) On the other hand, if each vertex has degree at least 6, we have

$$\sum_{v \in G} \deg(v) \geq \sum_{v \in G} 6 = 6V.$$

Consequently, we have $2E \geq 6V$, and therefore $E/3 \geq V$.

Similarly: notice that every face F of our planar graph must have at least three edges bounding it, because our faces are made out of edges in our graph. Also, if we sum over all faces the number of edges in each face, we get again twice the number of edges; this is because each edge is in exactly two faces. Therefore, we have

$$2E = \sum_{f \in G} \text{facedeg}(f) \geq \sum_{f \in G} 3 = 3F,$$

and therefore that $2E/3 \geq F$.

Therefore, we have

$$2 = V - E + F \leq E/3 - E + 2E/3 = 0,$$

which is clearly impossible. Therefore, we have a contradiction, and can conclude that our initial assumption — that all vertices have degree at least 6 — is false! $\square$

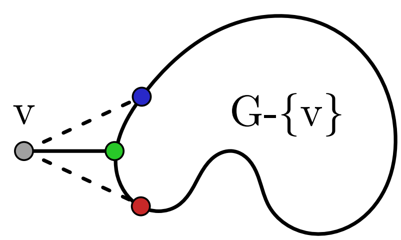
Proof. We proceed by contradiction. Assume not: that there are connected planar graphs on finitely many vertices that need at least 5 colors to be colored properly. Consequently, there must be some **smallest** connected planar graph G, in terms of the number of its vertices, that needs at least five colors to color its vertices! Pick such a graph G. Notice that if we remove any vertex v from G, we have a graph on a smaller number of vertices than G. Consequently, the graph $G \setminus \{v\}$ **can** be colored with four colors!

Let v be the vertex in G with degree at most 5. Delete v from G: this leaves us a graph that we can four-color. Do so.

Our goal is now the following: to add v back in and (by possibly changing the coloring of $G \setminus \{v\}$) give v one of our four colors, so that we have a four-coloring of G! This will prove that our initial assumption — that a G can exist that needs five colors — is false, and therefore prove our theorem.

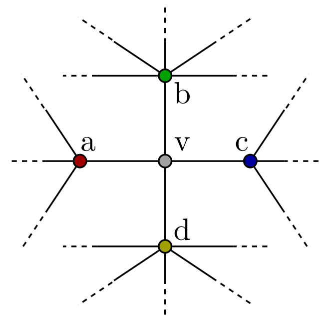
We proceed by cases, considering v's possible degrees.

1. v has degree 1, 2 or 3. In these cases, note that when we add v back in, it is adjacent to at most three colors! So there is some fourth color left over. Give v that color.



2. v has degree 4. In this case, there are two possibilities:

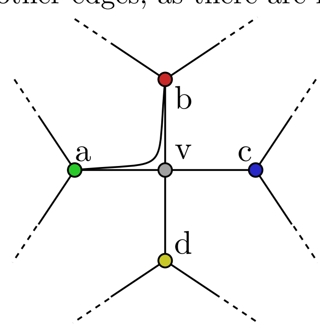
    - In the four neighbors a, b, c, d of v, some color is not used. In this case, we are in the same kind of situation as above: just color v with the color that doesn't show up in its neighbors?
    - In the four neighbors a, b, c, d of v, each color is used exactly once. So, up to the names of the colors, we are in the following situation:



First, notice that without losing any generality we may assume that there are edges $a \leftrightarrow b \leftrightarrow c \leftrightarrow d \leftrightarrow a$. To see why, notice the following:

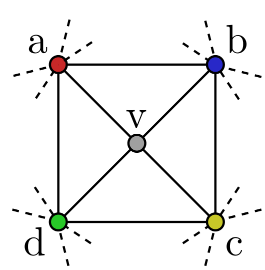
(a) Adding these edges does not change the assumed property that $\chi(G) \geq 5$, as extra edges only makes it "harder" for us to color a graph.

(b) We can always draw in these edges if they do not exist: for example, if the edge $\{a, b\}$ did not exist, we could add it in without breaking planarity by simply drawing a path that is "very close" to the two edges $\{a, v\}$ and $\{v, b\}$. This path will not cross other edges, as there are no other edges leaving v.



(c) Therefore, because we have preserved $\chi(G) \geq 5$ and G 's planarity, we can put these edges into G without changing any of our arguments thus far!

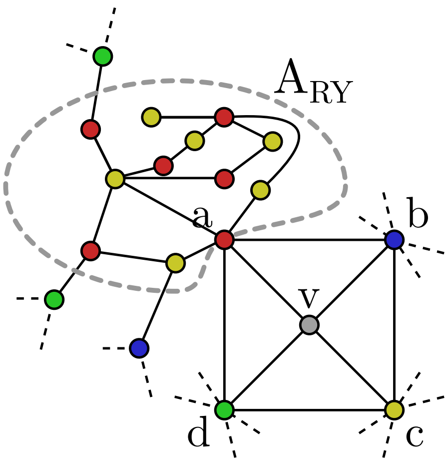
By the argument above, we can now assume our graph looks like the following:



Now, do the following: for any two colors C_1, C_2, let G_{C_1,C_2} denote the subgraph of G given by taking all of the vertices in G that are colored either C_1 or C_2, along with all of the edges that connect C_1 vertices to C_2 vertices.

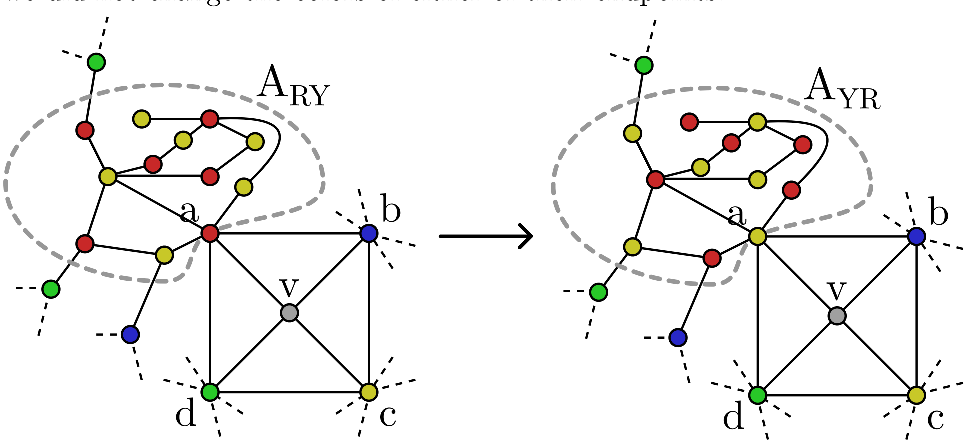
Look at the red-yellow subgraph G_{RY}. In this graph, there are two possibilities:

(a) There is no path from a to c in this graph. In other words, define A_{RY} as the subgraph of G_{RY} given by taking all of the G_{RY} vertices that have paths to a, along with all of the edges in our graph between such vertices.



Suppose that we "switch" the colors red and yellow in the subgraph A_{RY}. Does it create any issues with our coloring?

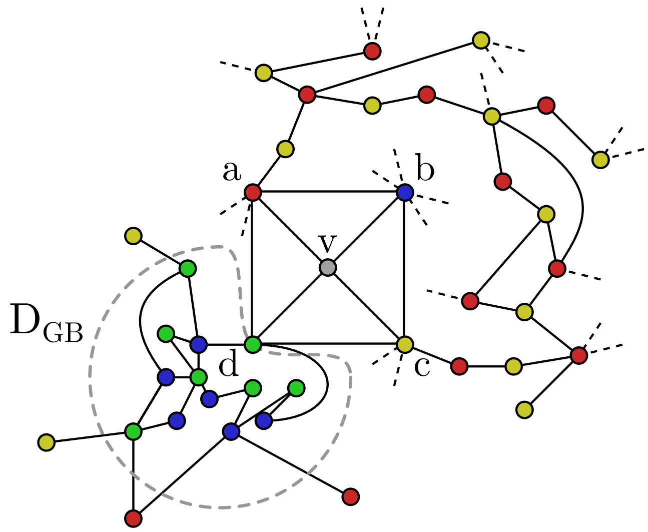
Let's check. No edge between two vertices in A_{RY} is broken (i.e. has both endpoints made the same color) by this process; before it had one red and one yellow endpoint, and now it has one yellow and one red endpoint. As well, no edge that involves no vertices in A_{RY} is broken by this process, as we did not change the colors of either of their endpoints!



Finally, consider any edge with one endpoint in A_{RY} and another endpoint not in A_{RY}. In order for this edge to have one endpoint in A_{RY} and another not in A_{RY}, one endpoint must be red or yellow (the endpoint in our set) and the other must be green or blue (the endpoint not in our set!) So if we switch red and yellow in A_{RY}, this edge is also not broken!

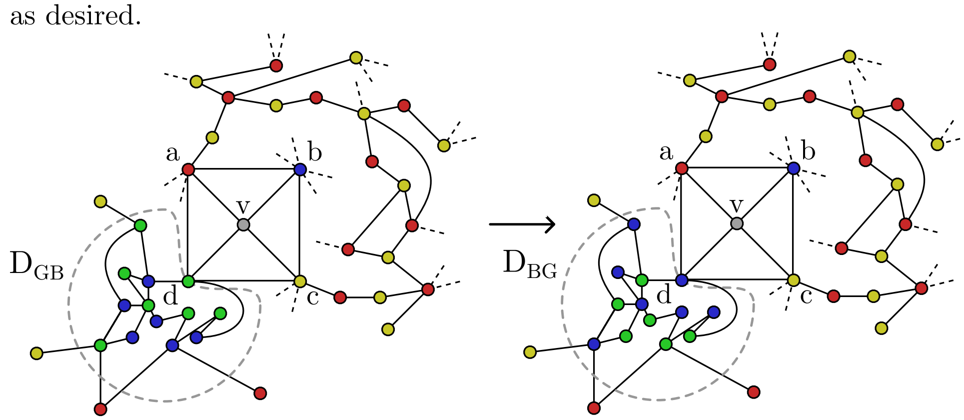
No edges are broken by this swap; therefore we still have a valid coloring. Furthermore, in this coloring, v has no neighbors that are red; so we can color v red and have a four-coloring of our entire graph G!

(b) Alternately, (a) does not happen. In this case, there is a path from a to c made entirely of red-yellow vertices linked by edges. In this case: look at the graph G_{GB}.

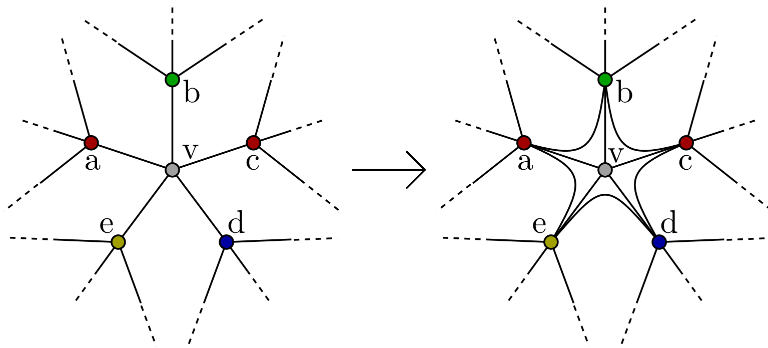


In particular, notice that there cannot be a path from b to d along green-blue edges, because our graph is planar and any such path would have to cross our red-yellow edges! Therefore, we can define D_{GB} to be the collection of all of the G_{GB} vertices that have paths to d, along with the edges in our graph between such vertices. As noted above, $d \notin B_{GB}$.

Now, switch the colors G and B in D_{GB}! This causes no conflicts, by exactly the same argument as above, and yields a graph where v has no green neighbor; therefore, we can give v the color green, and have a proper four-coloring as desired.



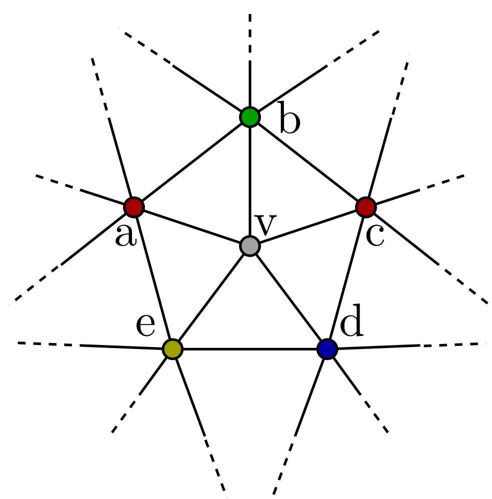
3. v has degree 5. Again, as before, we can assume that all four of the colors in our graph occur on G's neighbors, because if they do not we can simply give v whichever color is missing. Again, as before, we can assume that the neighbors of v are connected by the following pentagonal structure.:



This is because of the following:

- Adding edges to our graph will never make it easier to color a graph: all they do is give us more conditions on what vertices have to have different colors, which only makes coloring **harder**.
- Furthermore we can add these edges without breaking planarity by simply drawing them arbitrarily close to the v-edges.

Up to symmetry and colorings, then, we are in the following situation:



This is because we have to repeat one color (so it might as well be red,) we have to use all of the other colors (so we have green, blue and yellow in some order,) red cannot occur on two adjacent vertices (because there are edges between adjacent vertices,) and therefore up to rotation and flipping we have the above.

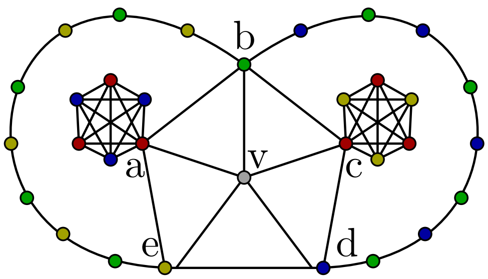
Do the following:

(a) First, look at the G_{GB} subgraph. Either the vertex b is not connected to d in this subgraph, in which case we can do the switching-trick that we discussed earlier. Otherwise, b **is** connected to d in G_{GB}, and we have a green-blue chain from b to d.

(b) Now, look at the G_{GY} subgraph. Similarly, either the vertex b is not connected to e in this subgraph, in which case we can do the switching-trick that we discussed earlier, or it is, and we have a green-yellow chain from b to e.

If we were able to switch in either of the two cases above, then v has only three colors amongst its neighbors, and we can color it with whatever color remains.

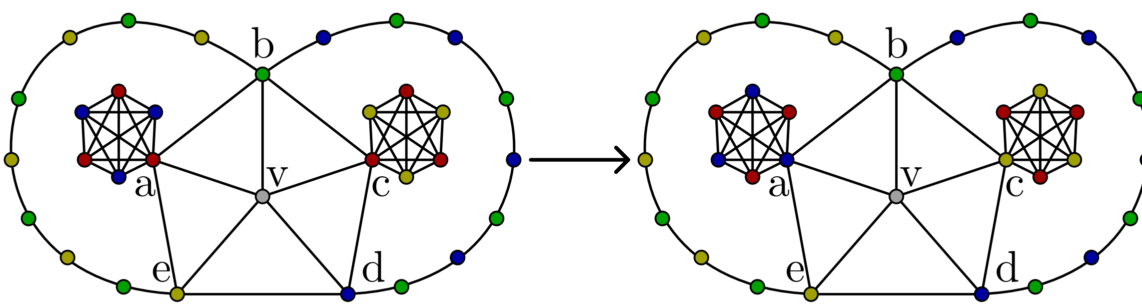
Otherwise, we are in the following case:



Do the following:

(a) First, look at the G_{RB} subgraph. Because of the green-yellow chain, the vertices a and d are not connected to each other. Therefore, we can switch red and blue in the a-connected part of this subgraph!

(b) Now, look at the G_{RY} subgraph. Because of the green-blue chain, the vertices c and e are not connected to each other. Therefore, we can switch red and yellow in the c-connected part of this subgraph!



This yields a graph where v has no red neighbor: consequently, we can color v red, which gives us a proper four-coloring! This proves our claim.

□

2 Plot Twist

This proof . . . is in fact Kempe's famous flawed proof of the four-color theorem, which stood for 11+ years before being disproven! In particular, it was **disproven**: i.e. the proof you've read above is **false**!

Guess what your HW is for today?