Math 7H: Honors Seminar | Professor: Padraic Bartlett

Minilecture 3: A (false) Proof of the Four-Color Theorem

Week 5 | *UCSB 2014*

In today's talk, we're going to study the Four-Color Theorem. In particular, we're going to consider a proof of the Four-Color Theorem, given by Kempe in 1879:

Theorem. (Kempe, 1879.) Take any map, which for our purposes is a way to partition the plane $\mathbb{R}^2$ into a collection of connected regions $R_1, \ldots R_n$ with continuous boundaries. There is some way to assign each region R_i to a color in the set $\{R, G, B, Y\}$, such that if two regions R_i, R_j are "touching" (i.e. they share some nonzero length of boundary between them,) then those two regions must receive different colors.

Notably, Kempe's proof is **wrong**. It stood for 11 years until someone noticed a flaw in it; after then, it took us until 1976 to find a new proof that did not have a flaw. This new proof has been widely criticized as being "inelegant" because of its need for computer-aided search, and of the near-impossibility for any human to read through the computer-verified parts. Later improvements have reduced the amount of cases a computer is needed for, but no fundamentally different proof to the computer-aided one is known.

So: why look at a false proof? Two reasons:

- The techniques used in this proof are widely used in the actual proof! In a loose sense, the proof we are going to explore below works by taking a specific vertex in a graph, and looking at its neighbors – i.e. all of the vertices one step away. In a certain sense, the correct proof of the four-color theorem consists of doing the same thing, but looking at both the vertex, its neighbors, and its neighbor's neighbors. This, with some careful rigor, creates a large bur finite number of cases to check, which a computer can do.
- It is valuable to see persuasive yet false things, and to practice disproving them. Part of developing as a mathematician is learning how to be critical of your own proofs and those of others!

Consequently, the HW set for this class is just one very simple problem: find the flaw in this paper's argument!

We start with some background. (As a hint: nothing in the background is false. The flaw occurs in the proof section.)

1 Background

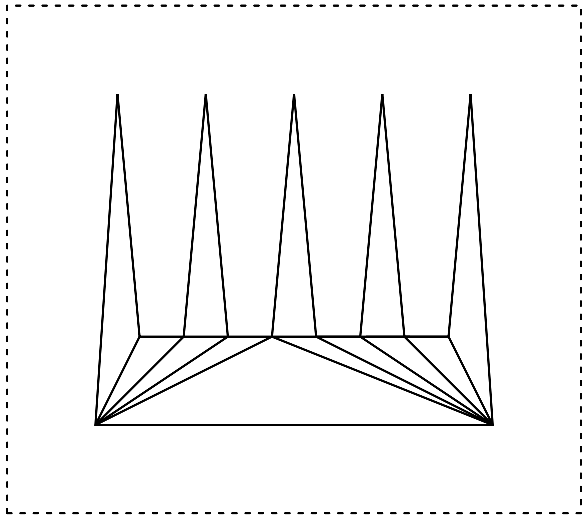
Definition. Take any map M. We can turn this into a graph as follows:

- Assign to each region R_i a vertex v_i.
- Connect v_i to v_j with an edge if the regions R_i, R_j are touching.

We call this graph the **dual graph** to M.

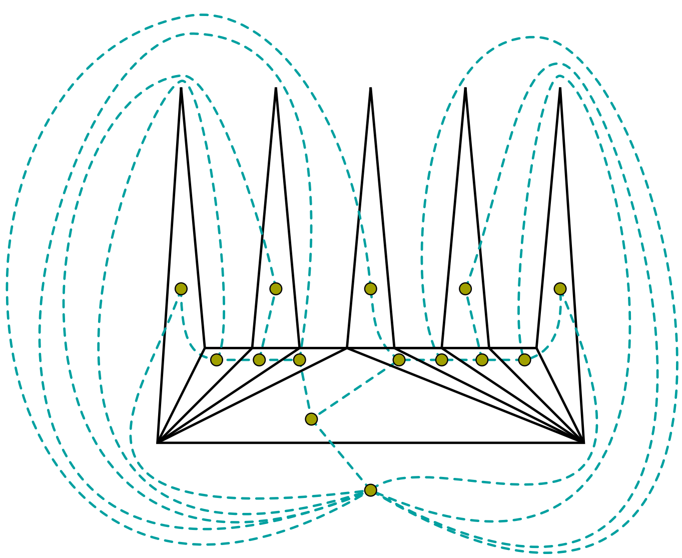
We give an example here:

Example. Consider the following map:



This map consists of 14 regions. If you count, you can see that the figure drawn consists thirteen triangles; as well, we have the “outer” region consisting of everything else left over, which forms a very strange 15-gon.

Now, take each region, and assign to it a vertex. As well, connect two regions sharing a border with an edge: this will give you the following graph, with edges given by the dashed teal lines and vertices given by the yellow dots:



Note how we have drawn the edges so that they connect two adjacent countries by traveling through the border that they share! This observation is useful to recall when thinking about our second definition:

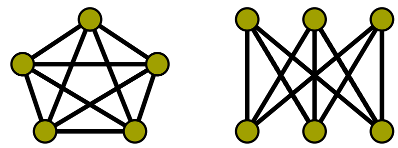
Definition. We say that a graph G is **planar** if we can draw it in the plane so that none of its edges intersect.

Given any planar graph G, we can make the following definition:

Definition. A **face** of G is any region of G bounded by the edges of G.

We will talk about planar graphs in more depth in a later talk, but the following two observations are valuable to make:

Observation. The graphs K_5 and $K_{3,3}$ are nonplanar.



This isn't obvious: try to prove it!

Observation. The dual graph to any map M is planar.

This is not too hard to prove; do it if you're interested!

The reason we care about this is that it gives us the following more graph-theoretic way to describe the four-color theorem:

Theorem. Take any planar graph on finitely many vertices. There is a way to assign each of its vertices one of the four colors $\{R, G, B, Y\}$ such that no edge in this graph has both endpoints colored the same color.

In general, this concept of **coloring** comes up all the time in graph theory! We give it a name here:

Definition. A graph G is called k**-colorable** if there is a collection of k distinct colors that we can map the vertices of G to, so that no edge in G has both endpoints colored the same color. Given a graph G, we define the **chromatic number** of G, $\chi(G)$, as the smallest number k such that G is k-colorable.

This gives us one last rephrasing of the four-color theorem:

Theorem. If G is a planar graph on finitely many vertices, then $\chi(G) \leq 4$.

...So. Before we can start Kempe's proof, we need one last bit of background, which is the concept of Euler characteristic:

Theorem. (Euler characteristic) Take any graph that has been drawn in $\mathbb{R}^2$ as a planar graph. Then, if V is the number of vertices, E is the number of edges, and F is the number of faces in this graph, we have the following relation:

$$V - E + F = 2.$$

Using this theorem, we can prove the following useful lemma, which is the only part of the Euler characteristic property that we need for our graph:

Lemma. Take any planar graph G. Then there is some vertex v in our graph with degree at most 5.

Proof. We proceed by contradiction. Assume that every vertex has at least degree 6; we will create a contradiction to the claim that $V - E + F = 2$.

First, consider the sum $\sum_{v \in G} \deg(v)$. On one hand, this is twice the number of edges in G: this is because each edge shows up twice in this sum (once for each endpoint v when we're calculating $\deg(v)$.) On the other hand, if each vertex has degree at least 6, we have

$$\sum_{v \in G} \deg(v) \geq \sum_{v \in G} 6 = 6V.$$

Consequently, we have $2E \geq 6V$, and therefore $E/3 \geq V$.

Similarly: notice that every face F of our planar graph must have at least three edges bounding it, because our faces are made out of edges in our graph. Also, if we sum over all faces the number of edges in each face, we get again twice the number of edges; this is because each edge is in exactly two faces. Therefore, we have

$$2E = \sum_{f \in G} \text{facedeg}(f) \geq \sum_{f \in G} 3 = 3F,$$

and therefore that $2E/3 \geq F$.

Therefore, we have

$$2 = V - E + F \leq E/3 - E + 2E/3 = 0,$$

which is clearly impossible. Therefore, we have a contradiction, and can conclude that our initial assumption — that all vertices have degree at least 6 — is false! □

2 Kempe's Proof

With this notation set up, Kempe's proof is actually fairly straightforward! We give it here. (For the HW, this is where you should look for a flaw.)

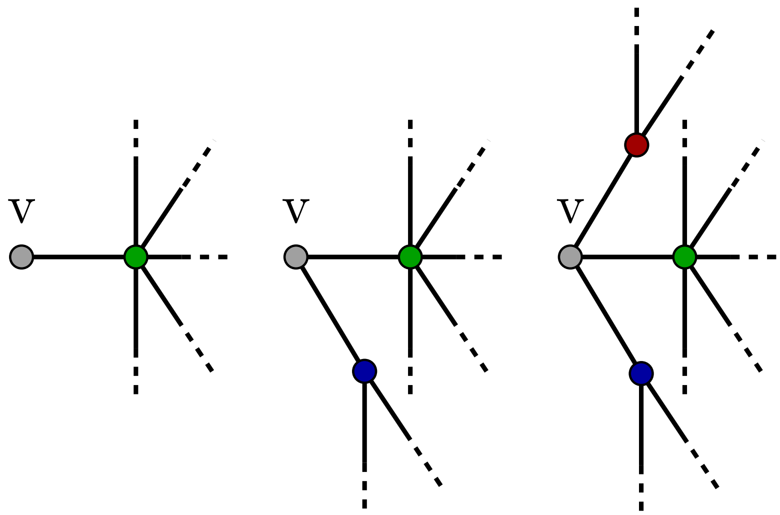
Proof. We proceed by contradiction. Assume not: that there are planar graphs on finitely many vertices that need at least 5 colors to be colored properly. Consequently, there must be some **smallest** planar graph G, in terms of the number of its vertices, that needs at least five colors to color its vertices! Pick such a graph G. Notice that if we remove any vertex v from G, we have a graph on a smaller number of vertices than G. Consequently, the graph $G \setminus \{v\}$ **can** be colored with four colors!

Let v be the vertex in G with degree at most 5. Delete v from G: this leaves us a graph that we can four-color. Do so.

Our goal is now the following: to add v back in and (by possibly changing the coloring of $G \setminus \{v\}$) give v one of our four colors, so that we have a four-coloring of G! This will prove that our initial assumption — that a G can exist that needs five colors — is false, and therefore prove our theorem.

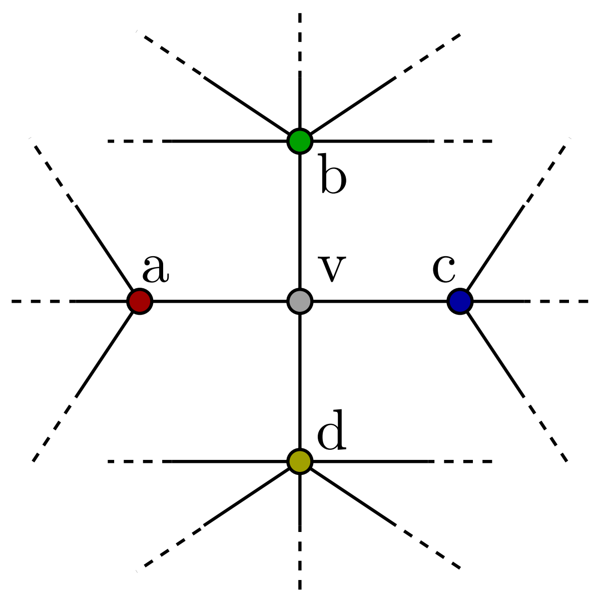
We proceed by cases, considering v's possible degrees:

1. v has degree 1, 2 or 3. In these cases, notice that when we add v back in, it is adjacent to at most three other colors! So there is some fourth color left over that we can assign to v. Do so.



2. v has degree 4. In this case, there are two possibilities:

- In the four neighbors a, b, c, d of v, some color is not used. In this case, we are in the same kind of situation as above: just color v with the color that doesn't show up in its neighbors?
- In the four neighbors a, b, c, d of v, each color is used exactly once. So, up to the names of the colors, we are in the following situation:

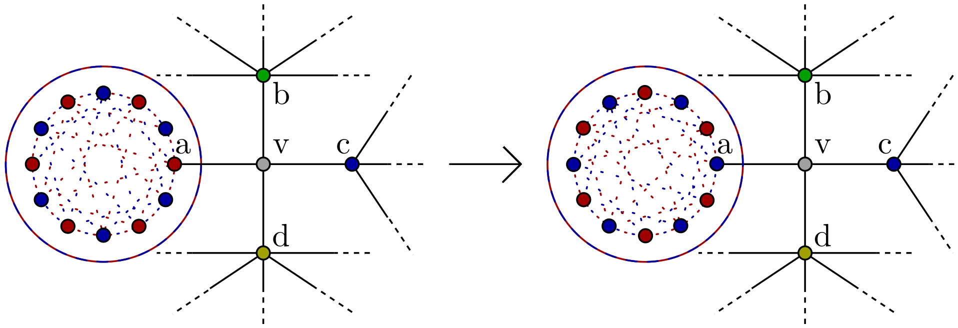


Do the following: for any two colors C_1, C_2, let G_{C_1,C_2} denote the subgraph of G given by taking all of the vertices in G that are colored either C_1 or C_2, along with all of the edges that connect C_1 vertices to C_2 vertices.

Look at the red-blue subgraph G_{RB}. In this graph, there are two possibilities:

(a) There is no path from a to c in this graph. In other words, define A_{RB} as the subgraph of G_{RB} given by taking all of the G_{RB} vertices that have paths to a, along with all of the edges in our graph between such vertices: we are dealing with the situation that $c \notin A_{RB}$.

Suppose that we "switch" the colors red and blue in this connected subgraph.

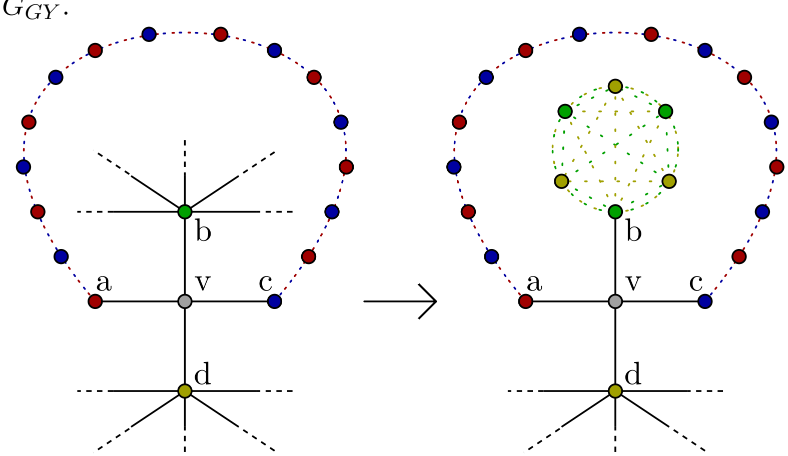


Does it create any issues with our coloring?

Let's check. No edge between two vertices in A_{RB} is broken (i.e. has both endpoints made the same color) by this process; before it had one red and one blue endpoint, and now it has one blue and one red endpoint. As well, no edge that involves no vertices in A_{RB} is broken by this process, as we did not change the colors of either of their endpoints! Finally, consider any edge with one endpoint in A_{RB} and another endpoint not in A_{RB}. In order for this edge to have one endpoint in A_{RB} and another not in A_{RB}, one endpoint must be red or blue (the endpoint in our set) and the other must be green or yellow (the endpoint not in our set!) So if we switch red and blue in A_{RB}, this edge is also not broken!

No edges are broken by this swap; therefore we still have a valid coloring. Furthermore, in this coloring, v has no neighbors that are red; so we can color v red and have a four-coloring of our entire graph G!

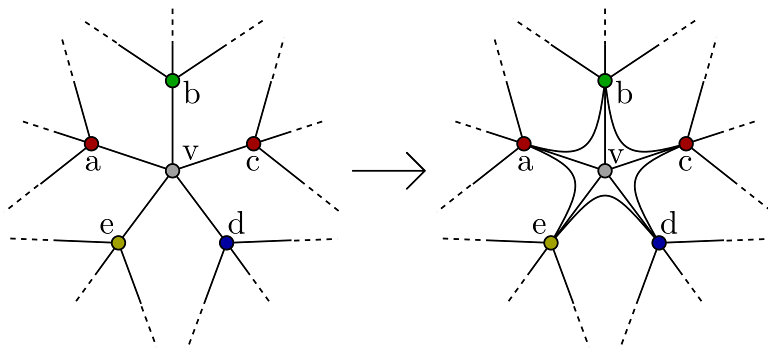
(b) Alternately, (a) does not happen. In this case, there is a path from a to c made entirely of red-blue vertices linked by edges. In this case: look at the graph G_{GY}.



In particular, notice that there cannot be a path from b to d along green-yellow edges, because our graph is planar and any such path would have to cross our red-blue edges. Therefore, we can define B_{GY} to be the collection of all of the G_{GY} vertices that have paths to b, along with the edges in our

graph between such vertices. As noted above, $d \notin B_{GY}$.
Switch the colors G and Y in C_{GY}! This causes no conflicts, by exactly the same argument as above, and yields a graph where v has no green neighbor; therefore, we can give v the color green, and have a proper four-coloring as desired.

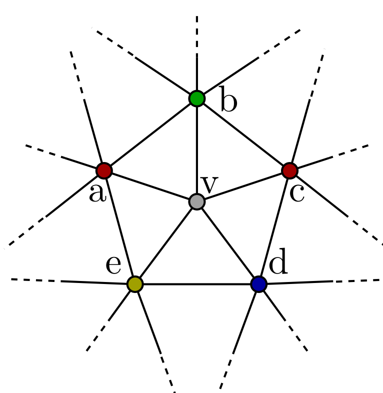
3. v has degree 5. Again, as before, we can assume that all four of the colors in our graph occur on G's neighbors, because if they do not we can simply give v whichever color is missing. Notice that we can actually assume that the neighbors of v are connected by the following pentagonal structure.



This is because of the following:

- Adding edges to our graph will never make it easier to color a graph: all they do is give us more conditions on what vertices have to have different colors, which only makes coloring **harder**.
- Furthermore we can add these edges without breaking planarity by simply drawing them arbitrarily close to the v-edges.

Up to symmetry and colorings, then, we are in the following situation:



This is because we have to repeat one color (so it might as well be red,) we have to use all of the other colors (so we have green, blue and yellow in some order,) red cannot occur on two adjacent vertices (because there are edges between adjacent vertices,) and therefore up to rotation and flipping we have the above.

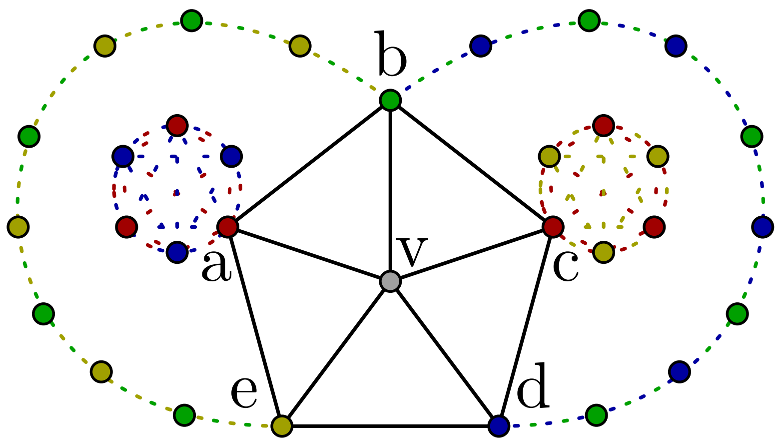
Do the following:

(a) First, look at the G_{BG} subgraph. Either the vertex b is not connected to d in this subgraph, in which case we can do the switching-trick that we discussed earlier. Otherwise, b **is** connected to d in G_{BG}, and we have a blue-green chain from b to d.

(b) Now, look at the G_{BY} subgraph. Similarly, either the vertex b is not connected to e in this subgraph, in which case we can do the switching-trick that we discussed earlier, or it is, and we have a blue-yellow chain from b to e.

If we were able to switch in either of the two cases above, then v has only three colors amongst its neighbors, and we can color it with whatever color remains.

Otherwise, we are in the following case:



Do the following:

(a) First, look at the G_{RG} subgraph. Because of the blue-yellow chain, the vertices a and d are not connected to each other. Therefore, we can switch red and green in the a-connected part of this subgraph!

(b) Now, look at the G_{RB} subgraph. Because of the green-yellow chain, the vertices c and e are not connected to each other. Therefore, we can switch red and blue in the c-connected part of this subgraph!

This yields a graph where v has no red neighbor: consequently, we can color v red, which gives us a proper four-coloring! This proves our claim.

□

So: there's a flaw somewhere in the past two pages. What is it?