| Math 7H: Honors Seminar | Professor: Padraic Bartlett |
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| Lecture 3: Induction | |
| *Week 3* | *UCSB 2014* |

# 1 Induction

Sometimes, in mathematics, we will want to prove the truth of some statement $P(n)$ that depends on some variable $n$. For example:

- $P(n)$ = "The sum of the first $n$ natural numbers is $\frac{n(n+1)}{2}$."
- $P(n)$ = "If $q \geq 2$, we have $n \leq q^n$.
- $P(n)$ = "Every polynomial of degree $n$ has at most $n$ roots."

For any fixed $n$, we can usually use our previously-established methods to prove the truth or falsity of the statement. However, sometimes we will want to prove that one of these statements holds for **every** value $n \in \mathbb{N}$. How can we do this?

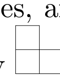
One method for proving such claims for every $n \in \mathbb{N}$ is the method of **mathematical induction**! Proofs by induction are somewhat more complicated than the previous two methods. We sketch their structure below:

- To start, we take our claim $P(n)$, that we want to prove holds for every $n \in \mathbb{N}$.
- The first step in our proof is the **base step**:in this step, we explicitly prove that the statement $P(1)$ holds, using normal proof methods.
- With this done, we move to the **induction step** of our proof: here, we prove the statement $P(n) \implies P(n+1)$, for every $n \in \mathbb{N}$. This is an implication; we will usually prove it directly by assuming that $P(n)$ holds and using this to conclude that $P(n+1)$ holds.

Once we've done these two steps, the principle of induction says that we've actually proven our claim for all $n \in \mathbb{N}$! The rigorous reason for this is the **well-ordering principle**, which we discussed in class; however, there are perhaps more intuitive ways to think about induction as well.

The way I usually think of inductive proofs is to think of toppling dominoes. Specifically, think of each of your $P(n)$ propositions as individual dominoes – one labeled $P(1)$, one labeled $P(2)$, one labeled $P(3)$, and so on/so forth. With our inductive step, we are insuring that all of our dominoes are *lined up* – in other words, that if one of them is true, that it will "knock over" whichever one comes after it and force it to be true as well! Then, we can think of the base step as "knocking over" the first domino; once we do that, the inductive step makes it so that all of the later dominoes also have to fall, and therefore that our proposition must be true for all $n$ (because all the dominoes fell!)

To illustrate how these kinds of proofs go, here's an example:

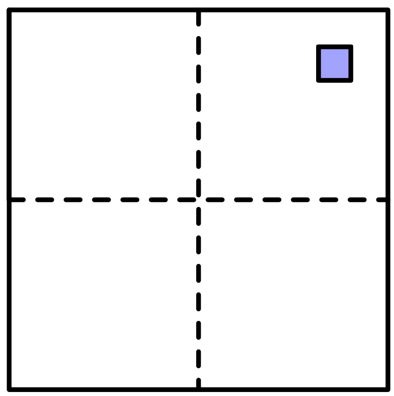
**Claim.** For any $n \in \mathbb{N}$, take a $2^n \times 2^n$ grid of unit squares, and remove one square from somewhere in your grid. The resulting grid can be tiled by  - shapes.

*Proof.* As suggested by the section title, we proceed by induction.

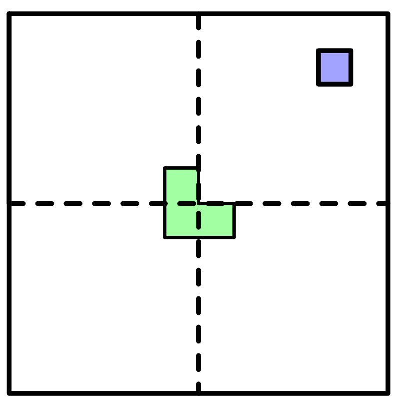
Base case: for $n = 1$, we simply have a $2 \times 2$ grid with one square punched out. As this *is* one of our three-square shapes, we are trivially done here.

Inductive step: Assume that we can do this for a $2^k \times 2^k$-grid without a square, for any $k \leq n$. We then want to prove that we can do this for a $2^{n+1} \times 2^{n+1}$ grid minus a square.

So: take any such grid, and divide it along the dashed indicated lines into four $2^n \times 2^n$ grids. By rotating our grid, make it so that the one missing square is in the upper-right hand corner, as shown below:



Take this grid, and carefully place down one thre-square shape as depicted in the picture below:



Now, look at each of the four $2^n \times 2^n$ squares in the above picture. They all are missing exactly one square: the upper-right hand one because of our original setup, and the other three because of our placed three-square-shape. Thus, by our inductive hypothesis, we know that all of these squares can also be tiled! Doing so then gives us a tiling of the whole shape; so we've created a tiling of the $2^{n+1} \times 2^{n+1}$ grid!

As this completes our inductive step, we are thus done with our proof by induction. $\square$

Let's look at another example of an inductive problem:

**Example.** Draw some straight lines in the plane. Notice that when we do this, we divide the plane up into regions bounded by these lines. What is the maximum number of regions we can divide the plane into with $n$ lines?

**Answer.** Again, take a moment to work out some base cases and figure out what's going on here!

Here's a few observations you're likely to have made:

1. No lines break the plane into one piece, as we've not split anything up! One line breaks the plane into two pieces; two lines breaks the plane into up to four pieces if those lines are not parallel; three lines can break the plane up into seven pieces if we are careful to not let all three lines intersect at the same place; and four lines can break the plane up into up to 11 pieces if we are again careful to not have any more than two lines intersect at any point, and also not have any parallel lines! In general, it looks like $n$ lines is giving us $1 + \frac{n(n+1)}{2}$ regions, given enough data and staring at things.

2. In general, it looks like the $n$-th line is adding at most $n$ new regions to our plane. To see why this might hold in general, consider the process of drawing any line.

   (a) If our line intersects any region, it divides that region into two pieces! This is the only way our line creates new regions.

   (b) Our line enters a region if and only if it crosses one of the lines that bounds that region.

   (c) As well, before our line crosses any other regions, it by default starts in some region already.

   (d) Therefore, the total number of times our line intersects other lines, plus one, is the total number of new regions created!

   (e) We can cross each other line at most once, as our lines are straight.

   (f) Therefore, if there are $n$ lines in existence, we can create at most $n + 1$ new regions by adding a $n + 1$-th line.

3. Furthermore, notice that it is always possible to draw such a line! To draw a line, we need to give two pieces of information:

   (a) Its slope needs to not be parallel to any other existing line's slope, to insure that it can intersect that line. There are only $n$ slopes currently used and infinitely many possibilities, so this is always possible.

   (b) Given a slope, we need to pick a $x$-intercept for our line. Furthermore, we want to do this so that our line does not intersect any other lines at places where multiple lines are already intersecting: this would make our line use up multiple "intersecting other line" instances, while only entering one region (which means we wouldn't get to $n+1$!) There are only finitely many such existing intersection points, and infinitely many choices of $x$-intercept; so we can also avoid all of these possibilities.

Therefore, it is possible to always draw a line that intersects other lines in $n$ places, and thus that creates $n+1$ new regions!

By the above, we have a rather nice recurrence relation: if $L_n$ is the total number of regions that we can divide the plane up into with $n$ lines, we have

$$L_{n+1} = (n+1) + L_n.$$

Using this, we can prove that our guess of $L_n = 1 + \frac{n(n+1)}{2}$ is right via induction:

Base case: We know $L_0 = 1 = 1 + \frac{0 \cdot 1}{2}$ from our case work.

Ind. step: Assume that $L_n = 1 + \frac{n(n+1)}{2}$ for each $n$ from 1 to $m$; we will seek to prove that $L_{m+1} = 1 + \frac{(m+1)(m+2)}{2}$. This is pretty quick: notice that

$$\begin{aligned}
L_{m+1} &= (m+1) + L_m, \text{ by our recurrence relation,} \\
&= (m+1) + \frac{m(m+1)}{2} + 1, \text{ by our inductive hypothesis,} \\
&= \frac{m^2 + m + 2m + 2}{2} + 1 \\
&= \frac{(m+1)(m+2)}{2} + 1, \text{ as claimed.}
\end{aligned}$$