Math 116	Professor: Padraic Bartlett

Midterm!

Due Friday, Week 5, at the start of class — *UCSB 2015*

This midterm has **two** sections!

1. **Non-Collaboration Section**. This section contains 4 quiz-styled problems, of which you will pick three. If you attempt more than three, only your first three problems will be graded. You are allowed to use your notes from class and the online class notes in this section; however, other resources (e.g. the internet, Mathematica, classmates) are off-limits. You get **three** hours to complete this section!

2. **Collaboration Section**. This section contains 4 homework-styled problems, of which you will pick two. If you attempt more than two, only your first two problems will be graded. You are allowed to use your notes from class, online class notes, Wikipedia, textbooks, Mathematica/Wolfram Alpha/etc, and can also collaborate with other people in the class. As with the HW, you must cite all of your collaborators, and prove any results that you want to use that haven't been proven in class. Also, all proofs/writeups/etc must be done in your own words. You get **nine** hours to complete this section!

The way time limits work is via the honor system, and as follows:

- On the second page of this midterm, you can find a table for logging start/stop times on your work.

- When you start work, write down when you start the problem.

- When you stop — i.e., to take a break, or go to sleep, or go for a run — write down when you stop.

- Work is somewhat subjective, but can be broadly construed as "thinking about and/or working on the problems." As long as you are being fairly reasonable and ethical here, you are likely doing the right thing!

You can send in questions by email or ask them at office hours; I will be a bit more cryptic than normal, but will still be helpful!

Good luck and have fun!

Non-Collaboration Section Timesheet

Time started	Time stopped

Collaboration Section Timesheet

Time started	Time stopped

1 Non-Collaboration Section

Do **three** of the **four** problems in this section!

1. Consider the sequence $\{a_n\}_{n=0}^{\infty}$, defined recursively as follows: $a_0 = 0, a_1 = 1$, and

$$a_n = \begin{cases} a_{n/2}, & n \text{ is even}, \\ a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil}, & n \text{ is odd}. \end{cases}$$

   For example, because 2 is even, we have that $a_2 = a_{2/2} = a_1 = 1$. Similarly, because 3 is odd, we have that $a_3 = a_{\lfloor 3/2 \rfloor} + a_{\lceil 3/2 \rceil} = a_1 + a_2 = 1 + 1 = 2$.

   (a) Prove that for every $n \in \mathbb{N}$, we have $a_{2^n} = 1$.

   (b) Prove that for every $n \in \mathbb{N}, k \leq 2^n$, we have the following equality:

$$a_{2^n+k} = a_{2^n-k} + a_k.$$

2. The Pell numbers are defined as follows: $p_0 = 0, p_1 = 1$, and $p_n = 2p_{n-1} + p_{n-2}$. Let $P(x) = \sum_{n=0}^{\infty} p_n x^n$ denote the generating function for the Pell numbers.

   (a) Show that

$$P(x) = \frac{x}{1 - 2x - x^2}.$$

   (b) Use this to find a closed form for p_n.

3. Call a natural number n "powerful" if it is of the form m^k for some $m \in \mathbb{N}, k \geq 2 \in \mathbb{N}$. How many non-powerful natural numbers are there that are less than 10,000? (For this problem, you are allowed to use a calculator.)

4. Take any two finite sets A, B such that $A \subseteq B$. Prove that

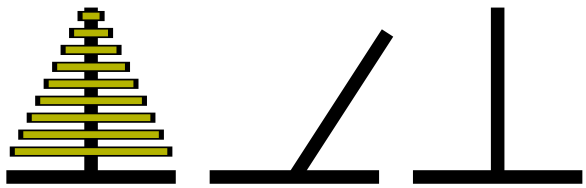
$$\sum_{A \subseteq K \subseteq B} (-1)^{|K \setminus A|} = \begin{cases} 1, & A = B, \\ 0, & A \neq B. \end{cases}$$

2 Collaboration Section

Do **two** of the four problems in this section! Each problem is on its own page.

1. Great news: for your birthday, you got a totally sweet Towers of Hanoi set!

   Bad news: in your mathematical joy, when you unwrapped your present you bent the middle peg pretty badly.



As a result, when you play with your Towers of Hanoi set, you can't really leave pegs on the middle tower for too long. To help with this, you've came up with a variation on the Towers of Hanoi rules. The normal Towers of Hanoi rules are as follows:

- Start with 3 rods A, B, C.
- On rod A, place n disks with radii $1, 2, \ldots n$, so that the disk with radius n is on the bottom, the disk with radius $n-1$ is on top of that disk, and so on/so forth.
- Your goal is to move all of the disks from the far-left rod A to the far-right rod C, subject to the following rules:
  - You can move only one disk at a time.
  - Each move consists of taking the top disk off of some rod and placing it on another rod.
  - You cannot place a disk D_i on top of any other disk D_j with radius smaller than D_i.

You've changed the rules above as follows:

- If you are moving disks from the broken rod B, you can move them all as one big stack (instead of one-by-one.) You still can't place bigger disks on top of smaller disks, but this should hopefully make it so we move disks away from B faster!

(a) Let h_n denote the smallest number of moves needed to move all of the disks from A to C in this new game. (Note: we assume $h_0 = 1$ here for calculational convenience.) Find a recurrence relation on the h_n's.

(b) Use this recurrence relation to find $H(x)$, the generating function for the $\{h_n\}_{n=0}^{\infty}$ sequence. (Hint: if you get this part right, you should have $H(x) = \frac{1-x+x^2}{1-2x+x^3}$.)

(c) Use this to find a closed form for h_n.

2. In class, we used the sieve methods to show that the number of permutations on $\{1, \ldots n\}$ with no fixed points was

$$n! \cdot \sum_{r=0}^{n} \frac{(-1)^r}{r!}.$$

(a) Generalize this: show that the number of permutations on $\{1, \ldots n\}$ with t fixed points is

$$\frac{n!}{t!} \cdot \sum_{r=0}^{n-t} \frac{(-1)^r}{r!}.$$

(b) Use the above to prove that for every $n \geq 4$, the number of permutations of $\{1, \ldots n\}$ with an even number of fixed points is greater than the number of permutations with an odd number of fixed points.

3. Congratulations: you've beat the Elite Four™and are now an official Pokémon Champion! Accordingly, you must now spend the next m days battling challengers, one each day, and you have n Pokémon. To start a Pokémon battle, you must pick one of your Pokémon to lead with.

   Suppose that over these m days, you want to pick each of your n Pokémon at least once to lead with. Let $A(m, n)$ denote the total number of different ways to make these choices.

   (a) Use inclusion-exclusion to prove

   $$A(m, n) = \sum_{k=0}^{n} \binom{n}{k} (n-k)^m (-1)^k.$$

   (b) As a corollary of the above, prove that if $m < n$, we have that

   $$\sum_{k=0}^{n} \binom{n}{k} (n-k)^m (-1)^k = 0.$$

4. You're writing a poem! Accordingly, because you're a mathematician, you've decided that you care very, very deeply about the structure of your poem. Namely, you've decided that your poem will have the following structure:

   - It will consist of n words. These words will be split into nonempty lines of consecutive words: i.e. we could split seven words into (four words + three words), (two words + one word + four words), (seven words), and many other such combinations.
   - In each line, exactly one word is the word "argyle." Any one of the words in a line can be "argyle."

   Let h_n denote the number of such structures (i.e. line breaks + placements of the word "argyle" in each line.) For example, $h_3 = 8$, as evidenced by the following structures:

   - □ argyle,
argyle
   - argyle □,
argyle
   - argyle,
□ argyle
   - argyle,
argyle □
   - argyle,
argyle,
argyle
   - argyle □ □
   - □ argyle □
   - □ □ argyle

   We find a closed form for h_n here. (Assume $h_0 = 1$ for computational convenience.)

   (a) Let w_n be the number of ways to take a line with n slots in it and pick one to be the word "argyle." Show that $w_n = n$, and therefore that if $W(x)$ is the generating function for the w_n's, then $W(x) = \frac{x}{(1-x)^2}$.

   (b) Let $h_n^{\{1\}}$ denote the number of poems on n words with our desired structure, that have exactly one line. If $H^{\{1\}}(x)$ is the generating function associated to the $h_n^{\{1\}}$'s, show that $H^{\{1\}}(x) = W(x)$.

   Similarly, let $h_n^{\{2\}}$ denote the number of poems on n words with our desired structure that have exactly two lines. If $H^{\{2\}}(x)$ is the generating function associated to the $h_n^{\{2\}}$'s, show that $H^{\{2\}}(x) = (W(x))^2$.

   (c) Generalize the above: let $h_n^{\{k\}}$ denote the number of poems on n words with our desired structure that have exactly k lines. If $H^{(k)}(x)$ is the generating function associated to the $h_n^{\{k\}}$'s, show that $H^{\{k\}}(x) = (W(x))^k$.

   (d) Let $H(x)$ be the generating function for our original h_n's. Show that

   $$H(x) = \sum_{k=0}^{\infty} H^{\{k\}}(x).$$

   (e) Use the above to find a closed form for $H(x)$, and from there find a closed form for h_n. (Hint: if you did this right, you should get $H(x) = 1 + \frac{x}{1-3x+x^2}$.)