Math 116	Professor: Padraic Bartlett

Homework 6: More Möbius Inversion; Group Actions

Due Friday, Week 8 — *UCSB 2015*

In this HW set, there are **two** sections: a **non-collaboration** section and a **collaboration** section. For the non-collaboration section, use only your notes/class notes, and don't work with others. For the collaboration section, work as normal!

1 Non-Collaboration Section

Do the one problem below!

1. Take a poset P. In this poset, we can define the concepts of least upper bounds and greatest lower bounds:

   - Given any set $S \subset P$, we say that z is an **upper bound** of S if $z \geq s$ for all $s \in S$. Similarly, we say that z is a **least upper bound** of S if z is an upper bound and for any other upper bound w, $z \leq w$.
   - Given any set $S \subset P$, we say that z is a **lower bound** of S if $z \leq s$ for all $s \in S$. Similarly, we say that z is a **greatest lower bound** of S if z is an lower bound and for any other lower bound w, $z \geq w$.

   We call a poset a **lattice** if for any $x, y \in P$, the set $\{x, y\}$ has a least upper bound and a greatest lower bound.

   (a) Show that the divisor posets and the Boolean posets we defined in class are lattices.

   (b) Is the poset from problem 2 on HW5 a lattice? Prove your claim.

   (c) Suppose that P is a lattice. For any $x, y \in P$, let $x \wedge y$ denote the greatest lower bound of x, y. Show that if $t \in P$ satisfies $x, y \geq t$, then $x \wedge y \geq t$ as well.

   (d) Similarly: suppose that P is a lattice. For any $x, y \in P$, let $x \vee y$ denote the least upper bound of x, y. Show that if $t \in P$ satisfies $x, y \leq t$, then $x \vee y \leq t$ as well.

2 Collaboration Section

Do **two** of the four problems below!

1. For any finite poset P, let $\mu(P)$ denote the maximum value of the Möbius function μ on the poset P. More generally, for any n, let $\mu(n)$ denote the maximum value of $\mu(P)$ amongst all finite posets P containing n elements.

   For each of the following claims about $\mu(n)$, decide whether it is true or false, and offer a proof for each claim.

(a) For all sufficiently large n, $\mu(n) > n$.

(b) For all sufficiently large n, $\mu(n) > n^2$.

(c) For all sufficiently large n, $\mu(n) > n^n$.

2. Suppose that L is a finite lattice with minimum element $\hat{0}$ and maximum element $\hat{1}$; that is, for any $x \in L$, we have $\hat{0} \leq x \leq \hat{1}$.

   Prove that for any $a \in L, a \neq \hat{1}$, we have

$$\sum_{x : x \wedge a = \hat{0}} \mu(x, \hat{1}) = 0.$$

   (Hint: for any $y, b \leq a$, let $N(y) = \sum_{x:x\wedge a = y} \mu(x, \hat{1})$ and $S(b) = \sum_{y:b\leq y \leq a} N(y)$. Show that $S(b) = 0$ for all $b \leq a$; then use Möbius inversion to prove the same thing for $N(y)$ for all $y \leq a$.)

3. Suppose you have a collection of seashells in k different colors, with as many as you could want of each color. You want to arrange them in a necklace like we did in class, where we consider two necklaces to be the same if one can be rotated around so that it becomes the other.

   (a) Prove that when n is prime, there are

$$\frac{k^n + k(n-1)}{n}$$

   many such necklaces.

   (b) Use the observation above to prove Fermat's little theorem: for any prime number k, we have $k^n \equiv k \mod n$.

4. Suppose that you have a regular tetrahedron and n different colors of paint In how many ways can you paint each of the sides of your tetrahedron a color, where we regard two tetrahedra as being the "same" if we can rotate one tetrahedron in space to get the other? (This oddly awkward YouTube video might be helpful for visualizing the rotations here; there are more than you may think!)

3 Extra-Credit Section

Solve any problem below for extra credit! (Warning: harder.)

1. For any n, let $\mu_L(n)$ denote the largest possible value of the Möbius function μ on any lattice poset L on n elements. Show that for sufficiently large n and any $\epsilon > 0$, $\mu_L(n) \geq n^{2-\epsilon}$.

2. (Open.) Prove that $\mu_L(n) \approx n^2$.