Math 116	Professor: Padraic Bartlett
Homework 5: Möbius Inversion	
Due Friday, Week 7	*UCSB 2015*

In this HW set, there are **two** sections: a **noncollaboration** section and a **collaboration** section. For the noncollaboration section, use only your notes/class notes, and don't work with others. For the collaboration section, work as normal!

1 Non-Collaboration Section

Do the one problem below!

1. Let P be a finite poset, and $f \in \mathbb{A}(P)$ be a function in the incidence algebra of P. Prove that the following properties are all equivalent:

   (i.) For any $x \in P$, we have $f(x,x) = 1$, and for any $x < y \in P$, we have
   $$\sum_{z : x \leq z \leq y} f(z,y) = 0.$$

   (ii.) For any $x \in P$, we have $f(x,x) = 1$, and for any $x < y \in P$, we have
   $$\sum_{z : x \leq z \leq y} f(x,z) = 0.$$

   (iii.) For any $x \in P$, we have $f(x,x) = 1$, and for any $x < y \in P$, we have
   $$\sum_{z : x < z \leq y} f(z,y) = -f(x,y).$$

   (iv.) For any $x \in P$, we have $f(x,x) = 1$, and for any $x < y \in P$, we have
   $$\sum_{z : x \leq z < y} f(x,z) = -f(x,y).$$

   (v.) f is μ, the Möbius function on P.

2 Collaboration Section

Do **two** of the four problems below!

1. Take any two finite posets P, Q. We can define the **direct product** $P \times Q$ of P and Q as the following partially ordered set:

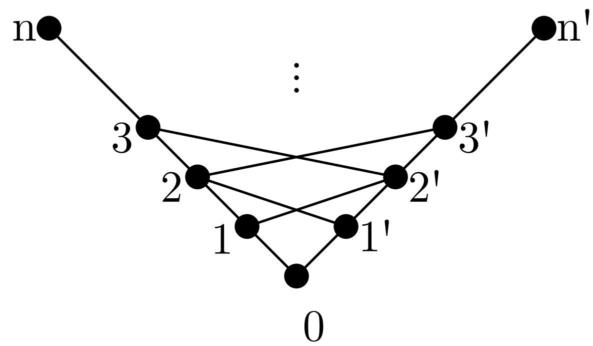
   - The elements of $P \times Q$ is the set $\{(p, q) \mid p \in P, q \in Q\}$.
   - Given any two elements $(p_1, q_1), (p_2, q_2) \in P \times Q$, we say that $(p_1, q_1) < (p_2, q_2)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$, with $(p_1, q_1) \neq (p_2, q_2)$.

   Let μ_P be the Möbius function of P, μ_Q be the Möbius function of Q, and $\mu_{P\times Q}$ be the Möbius function of $P \times Q$. Prove that for any $a, b \in P, c, d \in Q$, we have

   $$\mu_{P\times Q}((a, c), (b, d)) = \mu_P(a, b) \cdot \mu_Q(c, d).$$

   (Hint: try using some of the equivalent characterizations of the Möbius function that you've proven in the non-collab section!)

2. Consider the poset P visualized as follows:

   If you prefer symbols to pictures, P is simply a poset on the set $\{1, 2, 3, \ldots n\} \cup \{1', 2', 3', \ldots n'\} \cup \{0\}$. The relation $<$ on P is defined as follows: for any x, y in our set, we can say that $x < y$ in our set precisely whenever $x < y$ as a pair of natural numbers (i.e. $2 < 3', 4' < 16, 3 < 5, 7 < 12$ are all true in our poset!)

   Find a closed form for the Möbius function μ of this poset P.

3. In our inclusion-exclusion lectures from week 4, we defined the **Euler φ-function** as follows:

   $$\varphi(n) = \text{ the number of integers in } \{1, \ldots n\} \text{ that are relatively prime to } n.$$

   (a) Show that for any $n \in \mathbb{N}$,

   $$n = \sum_{d\in\mathbb{N}:d|n} \varphi(d).$$

(b) Use Möbius inversion on the above to prove that for any $n \in \mathbb{N}$,

$$\frac{\varphi(n)}{n} = \sum_{d \in \mathbb{N} : d | n} \frac{\mu(1, d)}{d}.$$

4. Prove the following "multiplicative" version of the Möbius inversion theorem: suppose that P is a finite poset with a unique minimal element 0. Let f be any function from P to $\mathbb{R}^+$; using f, create the function $g : P \to \mathbb{R}^+$ by setting

$$g(a) = \prod_{x \leq a} f(x).$$

Prove that

$$f(a) = \prod_{x \leq a} (g(x))^{\mu(x,a)}.$$