Homework 4: Posets, Convolution, and Möbius Functions

Due Friday, Week 6 *UCSB 2015*

In this HW set, there are **two** sections: a **noncollaboration** section and a **collaboration** section. For the noncollaboration section, use only your notes/class notes, and don't work with others. For the collaboration section, work as normal!

1 Non-Collaboration Section

Do the one problem below!

1. Take any finite poset $P = (X, <)$ on $|X| = n$ elements. An **linear extension** of P is any bijective map $f : P \to \{1, 2, \ldots n\}$ such that if $x < y \in P$, then $f(x) < f(y)$.

   Prove that any finite poset has a linear extension.

2 Collaboration Section

Do **two** of the four problems below!

1. Consider the convolution $f * g$, the **convolution** of any two functions $f, g \in \mathbb{A}(P)$:

$$(f * g)(x, y) = \sum_{z : x \leq z \leq y} f(x, z) \cdot g(z, y).$$

   (a) Prove that convolution is **associative**: that is, for any $f, g, h \in \mathbb{A}(P)$, prove that $f * (g * h) = (f * g) * h$.

   (b) Is convolution **commutative**? That is: for any poset P and $f, g \in \mathbb{A}(P)$, can you prove that $f * g = g * f$? Or can you find a poset P and functions $f, g \in \mathbb{A}(P)$ such that $f * g \neq g * f$?

2. In class on Monday / in the notes online now, we proved the following theorem:

   Theorem. Let P be any poset, and let r be any function $P \to \mathbb{R}$. Suppose that P has a unique minimal element: that is, there is some $m \in P$ such that for all $x \in P, m < x$. Define the function $s : P \to \mathbb{R}$ as follows: for any $a \in P$, set

$$s(a) = \sum_{x \leq a} r(x).$$

   Then we can "invert" the formula above: that is, for any $a \in P$, we have

$$r(a) = \sum_{x \leq a} s(x)\mu(x, a).$$

There is a "flipped" version of this theorem, where we assume our poset has a unique maximal element:

Theorem. Let P be any poset with a unique maximal element: that is, there is some $M \in P$ such that for all $x \in P, M > x$. Let r be any function $P \to \mathbb{R}$.

Define the function $s : P \to \mathbb{R}$ as follows: for any $a \in P$, set

$$s(a) = \sum_{x \geq a} r(x).$$

Then we can "invert" the formula above: that is, for any $a \in P$, we have

$$r(a) = \sum_{x \geq a} s(x)\mu(a, x).$$

Prove this theorem!

3. Given a poset P, a **chain** in P is any subset $C \subseteq P$ such that any two elements in C are comparable: that is, for any $x, y \in C$, we either have $x < y, x = y$ or $x > y$.

   Take any finite poset P. Let δ be the Kronecker delta function and ζ be the zeta function on P, as defined in class. For any $a, b \in P$, define the function η as follows:

$$\eta(a, b) = \zeta(a, b) - \delta(a, b).$$

   Show that $\eta^k(a, b)$ is equal to the number of chains of length k whose smallest element is a and largest element is b. (The **length** of a chain is the number of elements past the first in the chain: that is, the length of the chain $x < y$ is 1, the length of the chain $x < y < z$ is 2, the length of the chain $x < y < z < w$ is 3...)

4. Take any finite poset P on n elements, and any linear extension $l : P \to \{1, \ldots n\}$ of P. Use this linear extension to label the elements of P as $\{x_1, \ldots x_n\}$, where $l(x_i) = i$.

   Given any map $f \in \mathbb{A}(P)$, define the $n \times n$ matrix A_f as follows:

$$A_f(i, j) = f(x_i, x_j).$$

   Show that for any two maps $f, g \in \mathbb{A}(P)$,

$$A_f \cdot A_g = A_{f*g}.$$

   (In other words: the definition of convolution we're using here actually comes from matrix multiplication!)