Math/CS 120: Intro. to Math — Professor: Padraic Bartlett

Lecture 7: π, e, Transcendence and Irrationality

Week 8 — *UCSB 2014*

> Hold on. You have to slow down. You're losing it. You have to take a breath. Listen to yourself. You're connecting a computer bug I had with a computer bug you might have had and some religious hogwash. You want to find the number 216 in the world, you will be able to find it everywhere. 216 steps from a mere street corner to your front door. 216 seconds you spend riding on the elevator. When your mind becomes obsessed with anything, you will filter everything else out and find that thing everywhere.
>
> Sol Robeson, π

These notes are from the day-before-Thanksgiving lecture I gave on π, e, transcendence, and irrationality! The original plan for this talk was to prove that π is transcendental (and thus not constructible!), but on further review that proof seemed to be a bit too awful/long for this class! Instead, we put together two smaller results, the techniques for which you can extrapolate to proving π is transcendental!

For this talk, recall the following definitions:

Definition. A number $r \in \mathbb{R}$ is called **rational** if we can find integers $a, b \in \mathbb{Z}$, $b \neq 0$, such that $r = \frac{a}{b}$.

A number $r \in \mathbb{R}$ is called **algebraic** if we can find some natural number m and constants $a_0, \ldots a_m \in \mathbb{Z}$ such that $a_0 + a_1 r + \ldots a_m r^m = 0$. In other words, r is the root of some integer-coefficient polynomial $p(x) = a_0 + a_1 x + \ldots a_m x^m$.

A number $r \in \mathbb{R}$ is called **transcendental** if it is not algebraic.

Theorem. π^2 is irrational.

Proof. This proof is weird! In particular, unlike many of our other proofs, some of the things we do here will seem remarkably unmotivated and arbitrary. As a rough idea, though, here's our motivation:

1. One nice characterization of π is that it is the smallest positive root of $\sin(x)$.
2. $\sin(x)$ is a function that is well-behaved when we apply calculus techniques like derivation and integration to it!
3. Therefore, in theory, we should be able to make some sort of integral expression that relates π to $\sin(x)$.
4. If we proceed by contradiction and assume π is rational, then this integral should be able to be forced to be an integer, or some integer-like thing.

5. Conversely, because this integral corresponds to a number that we actually know is irrational, if we're clever we will be able to force it to be positive + arbitrarily small by using that irrationality somehow.

6. There are no positive arbitrarily small integers; so this should be a contradiction!

How we do this is something that you should take number theory /analysis classes to see! For now, however, let's just do it. Suppose for contradiction that π is rational; write $\pi^2 = a/b$. Suppose that via some *deus ex machina* you were told to consider the following two functions:

$$\begin{aligned} f(x) &= \frac{x^n(1-x)^n}{n!}, \\ F(x) &= b^n\left(\pi^{2n}f(x) - \pi^{2n-2}f''(x) + \pi^{2n-4}f^{(4)}(x) - \ldots + (-1)^n\pi^0 f^{(2n)}(x)\right). \end{aligned}$$

n is undefined here for right now, but will eventually be some "sufficiently large" number. Make the following observations about these two functions:

1. $f(x)$ **is small.** Specifically, notice that on the interval $[0,1]$, the two functions $x, (1-x)$ are bounded below by 0 and above by, and therefore that
$$0 < f(x) = \frac{x^n(1-x)^n}{n!} < \frac{1}{n!}$$

2. For $k < n$, $\frac{d^k}{dx^k}(f(x))$ is 0 at $x = 0, 1$. This is not too hard to see. Think about taking derivatives of $f(x)$: by the product rule, for any k the derivatives of $\frac{d^k}{dx^k}\left(\frac{x^n(1-x)^n}{n!}\right)$ will just look like a sum of a bunch of terms of the form
$$\frac{\text{constants}}{n!}x^{n-l_1}(1-x)^{n-l_2},$$
for $l_1 + l_2 = k$. In particular, if $k < n$, then these are both nonzero powers, and thus when we plug in $x = 0, 1$ we get 0, as $(x)(1-x) = 0$ for $x = 0, 1$.

3. Similarly, for $k > 2n$, the derivative $\frac{d^k}{dx^k}(f(x))$ is identically 0, because $f(x)$ is a degree-$2n$ polynomial, and we just took more than $2n$ derivatives!

4. Now consider any $n \le k \le 2n$. Here, the only terms in the derivative that are possibly nonzero at $x = 0, 1$ occur when we've taken either n derivatives of x^n and plugged in $x = 0$, or n derivatives of $(1-x)^n$ and plugged in $x = 1$. In either case, note that taking n derivatives of these two expressions multiplies our expressions by $n!$, as $\frac{d^n}{dx^n}x^n = n!$, $\frac{d^n}{dx^n}(1-x)^n = (-1)^n(n!)$.

   Consequently, we can note that for these values, $f^k(x)$ is an **integer** for $x = 0, 1$!

5. Now, consider $F(0)$ and $F(1)$. By our earlier observations about when the derivatives

of $f(x)$ are 0 when we plug in $x = 0, 1$, we can see that for $x = 0, 1$ we have

$$\begin{aligned}
F(x) &= b^n \left(\sum_{k=0}^{n} \pi^{2n-2k} f^{(2k)}(x)(-1)^k \right) \\
&= b^n \left(\left(\sum_{k=0}^{\lfloor n/2 \rfloor} \pi^{2n-2k} 0(-1)^k \right) + \left(\sum_{k=\lceil n/2 \rceil}^{n} \pi^{2n-2k} f^{(2k)}(x)(-1)^k \right) \right) \\
&= \sum_{k=\lceil n/2 \rceil}^{n} b^n \pi^{2n-2k} f^{(2k)}(x)(-1)^k.
\end{aligned}$$

What do we know about the terms in this sum? Well:

- The $f^{(2k)}(x)$'s are all integers.
- The $(-1)^k$'s are all integers.
- The $b^n \pi^{2n-2k}$'s are all integers; you can see this by writing $\pi = \frac{a}{b}$, and therefore that $b^n \frac{a^{2n-2k}}{b^{2n-2k}}$ is also an integer, because $2n - 2k \leq n$ above!

Therefore, the entire sum is an integer! That's cool.

6. Now, look at the expression

$$\begin{aligned}
&\frac{d}{dx} \left(F'(x) \sin(\pi x) - \pi F(x) \cos(\pi x) \right) \\
=&F''(x) \sin(\pi x) + \pi F'(x) \cos(\pi x) - \pi F'(x) \cos(\pi x) + \pi^2 F''(x) \sin(pix) \\
=&(F''(x) + \pi^2 F(x)) \sin(\pi x).
\end{aligned}$$

If you look at what happens to $F(x)$ as you take two derivatives of it, you can see that it basically shifts all of the $f^{(k)}(x)$ objects one spot over; if we scale by π^{-2} then the π-terms also line up, with the only difference being the signs of the terms!

In particular, this means that if we look at $F''(x) + \pi^2 F(x))$, all of the terms in the sum for $F(x)$ cancel out except for the first and last! Therefore, we have

$$\begin{aligned}
\frac{d}{dx} \left(F'(x) \sin(\pi x) - \pi F(x) \cos(\pi x) \right) =&(F''(x) + \pi^2 F(x)) \sin(\pi x) \\
=&b^n (f^{2n+2}(x)(-1)^n \pi^0 + \pi^{2n+2} f(x)) \sin(\pi x).
\end{aligned}$$

But we showed in 3 that derivatives of $f(x)$ past $2n$ are just 0; so we can ignore the first term in this sum, and get

$$\begin{aligned}
\frac{d}{dx} \left(F'(x) \sin(\pi x) - \pi F(x) \cos(\pi x) \right) &= b^n (\pi^{2n+2} f(x)) \sin(\pi x) \\
&= b^n (\frac{a^{n+1}}{b^{n+1}} f(x)) \sin(\pi x) \\
&= \frac{a^{n+1}}{b} f(x) \sin(\pi x).
\end{aligned}$$

7. Nearly done! Now, notice that on one hand, by the Fundamental Theorem of Calculus, you know that integration is just antidifferentiation: therefore,

$$\begin{aligned}
&\frac{1}{\pi}\int_0^1 \frac{d}{dx}\left(F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x)\right)dx \\
=&\frac{1}{\pi}\left(F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x)\right)\Bigg|_{x=0}^{1} \\
=&\frac{1}{\pi}\left(F'(0)\sin(\pi) - F'(1)\sin(\pi) - \pi F(0)\cos(0) + \pi F(1)\cos(\pi)\right) \\
=&-F(0) + F(1),
\end{aligned}$$

which is an integer as proven before!

8. On the other hand, notice that you can approximate this integral as well: if we write

$$\begin{aligned}
&\frac{1}{\pi}\int_0^1 \frac{d}{dx}\left(F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x)\right)dx \\
=&\int_0^1 a^{n+2} f(x)\sin(\pi x)dx,
\end{aligned}$$

we have by our first observation that

$$0 < \int_0^1 a^{n+2} f(x)\sin(\pi x)dx < \int_0^1 \frac{a^{n+2}}{n!}dx = \frac{a^{n+2}}{n!}.$$

For very large values of n, this RHS bound is less than 1, because a^{n+2} grows much slower than $n!$; to see why, note that if we increase n to $n+1$, the top grows by a factor of a and the bottom grows by a factor of $(n+1)$. Consequently, for very big values of n, the bottom is growing much faster than the top (i.e. if $n > 2a$, then each progressive increase of n by 1 shrinks the entire ratio $\frac{a^{n+2}}{n!}$ by a multiple of 1/2!)

9. So what have we shown? On one hand, this integral is an integer; on the other hand, it is between 0 and 1! This is a contradiction; therefore our initial assumption, that π^2 was rational, must be false!

$\square$

We can use similar techniques to prove that e is transcendental, as well!

Theorem. e is transcendental.

Proof. We proceed by the same blueprint of "pick out really weird functions related to nice properties of the constant (here, that e^x is a fun function to integrate,) and make an integral that on one hand is an integer and on the other hand is tiny.

Suppose for contradiction that e is rational. Then, we can find integers $a_0, \ldots a_m$ such that

$$a_0 + a_1 e + \ldots a_m e^m = 0.$$

As before, suppose that via some *deus ex machina* you were told to consider the following two functions:

$$f(x) = \frac{x^{p-1}(x-1)^p(x-2)^p \dots (x-m)^p}{(p-1)!},$$
$$F(x) = f(x) + f'(x) + f''(x) + \dots f^{(mp+p-1)}(x)$$

The m here is the same as from the algebraic expression for e above! p is not yet defined, but will be some "sufficiently large" prime.

Make the following observations about these two functions:

1. $f(x)$ **is small.** Specifically, notice that on the interval $[0, m]$, all of the functions $(x-k)$ are bounded above by m. There are $p-1+p+p+\dots+p = p-1+mp$ many such functions; therefore, we have the upper bound

$$|f(x)| = \frac{m^{mp+p-1}}{(p-1)!}.$$

2. For $k < p$, $\frac{d^k}{dx^k}(f(j))$ is 0 for any $j \in \{1, \dots m\}$, and for $k < p-1$ $\frac{d^k}{dx^k}(f(0))$ is 0. This is just like before: to get $f(j) \neq 0$ for any $j \in \{0, 1, \dots m\}$, you need to somehow get rid of the $(x-j)$-terms in $f(x)$. There are p-many such terms for $j \neq 0$ and $p-1$-many such terms for $j = 0$; so if we've taken less than that many derivatives, we will get 0.

3. Similarly, for $k > mp + p - 1$, the derivative $\frac{d^k}{dx^k}(f(x))$ is identically 0, because $f(x)$ is a degree-$mp + p - 1$ polynomial!

4. Actually, notice something stronger; for any $k \geq p$, $j \in \{0, 1 \dots p\}$, we have that $\frac{d^k}{dx^k}(f(j))$ is congruent to 0 modulo p. This is because if you take at least p derivatives of $f(x)$, you have to have both

   - taken at least one derivative of a $(x-l)^p$ term for some $l \neq 0$, and thereby gotten a multiple of p, while
   - if $\frac{d^k}{dx^k}(f(j))$ is not giving you 0, you must have taken at least $p-1$ derivatives of the $(x-j)^p$ term (or $p-1$ derivatives of the x^{p-1} term, if $j = 0$.) Taking these derivatives scales our object by $(p-1)!$, which gets rid of the denominator in $f(j)$.

   Therefore we are getting integers that are multiples of p!

5. So, the only possible derivative that may not be a multiple of p is for $k = p-1, j = 0$; that is, $f^{p-1}(0)$. This is in fact not a multiple of p! To see this, simply notice that the only nonzero term of this derivative when 0 is plugged in is just the one where we applied all $p-1$ derivatives to x^{p-1}, where we get

$$f^{p-1}(0) = (0-1)^p(0-2)^p \dots (0-m)^p.$$

   If we pick p to be a prime bigger than m, the RHS has no multiples of p, and thus is not zero mod p!

6. Now consider any $n \leq k \leq 2n$. Here, the only terms in the derivative that are possibly nonzero at $x = 0, 1$ occur when we've taken either n derivatives of x^n and plugged in $x = 0$, or n derivatives of $(1 - x)^n$ and plugged in $x = 1$. In either case, note that taking n derivatives of these two expressions multiplies our expressions by $n!$, as $\frac{d^n}{dx^n} x^n = n!$, $\frac{d^n}{dx^n}(1 - x)^n = (-1)^n(n!)$.

   Consequently, we can note that for these values, $f^k(x)$ is an **integer** for $x = 0, 1$!

   So, like before, we understand the derivatives of $f(x)$.

7. Now, like before, we consider a useful expression in terms of $F(x)$ that we'll want to integrate. Consider

$$\frac{d}{dx}\left(e^{-x}F(x)\right) = e^{-x}(F'(x) - F(x))$$

   Look at what happens to $F(x)$ when you take a derivative of it: it shifts exactly one spot over! Therefore, if we have this shifted difference, we again get that all of the terms cancel out except for the first term, $f(x)$, and the last term, $f^{(mp+p)}(x)$. But this last term is 0 by our observation 3 earlier; so we just have

$$\frac{d}{dx}\left(e^{-x}F(x)\right) = -e^{-x}f(x).$$

8. Let's do the same trick as before, where we interpret integrals of the above expression in two ways, depending on whether we antidifferentiate or just bound our expressions! In particular, look at

$$\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)dx.$$

   If we antidifferentiate, we get

$$\begin{aligned}\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)dx =& \sum_{j=0}^{m} a_j e^j \left(e^{-x}F(x)\right)\Big|_{x=0}^{j} \\ =& \left(\sum_{j=0}^{m} a_j e^j F(0)\right) - \left(\sum_{j=0}^{m} a_j F(j)\right) \\ =& F(0)\left(\sum_{j=0}^{m} a_j e^j\right) - \left(\sum_{j=0}^{m} a_j F(j)\right)\end{aligned}$$

   Notice that the first part of the RHS is 0, because it is just the algebraic expression that has e as a root! Therefore we only have the second part: that is,

$$\begin{aligned}\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x) dx &= -\sum_{j=0}^{m} a_j F(j) \\ &= -\sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j f^{(i)}(j),\end{aligned}$$

if we plug in what $F(j)$ is by definition.

But what is this RHS? It is an integer that is not 0 mod p, by our earlier study of the $f^{(i)}(j)$'s!

9. Nearly done! Now we just need to approximate this integral as well. To do this, write

$$\left| \sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x) dx \right| < \sum_{j=0}^{m} a_j e^j j \cdot \frac{m^{mp+p-1}}{(p-1)!} < \frac{a_{max} e^m m^{(m+1)p}}{(p-1)!}.$$

Again, notice that for large p, constants^p $<<$ $(p-1)!$ So again, this is arbitrarily small.

10. In conclusion: what have we shown? On one hand, this integral is an integer that is not zero mod p. On the other hand, it is arbitarily small! This is a contradiction; therefore our initial assumption, that e was transcendental, must be false!

$\square$

Math!