Math/CS 120: Intro. to Math | Professor: Padraic Bartlett

Lecture 9: Field Extensions

Week 9 | *UCSB 2014*

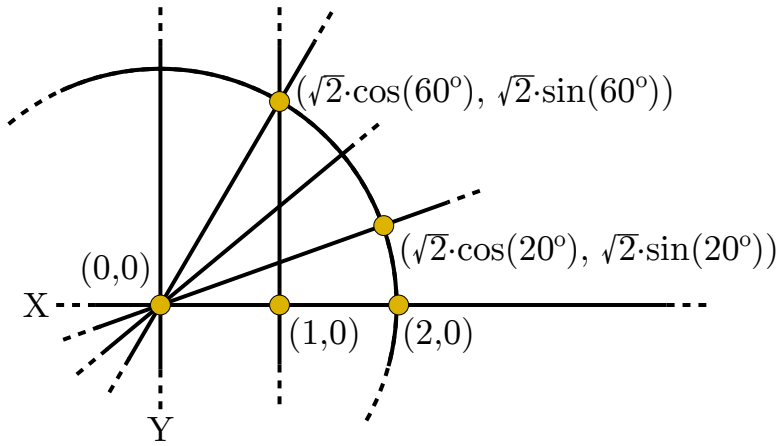
> In any field, find the strangest thing and then explore it.
>
> John Archibald Wheeler, physicist

1 Field Extensions

We ended our constructible numbers lectures with the following classical questions:

- **Doubling the Cube**: Can you construct a number x such that the volume of the cube with side length x is 2? In other words, can you construct $\sqrt[3]{2}$?
- **Trisecting the Angle**: Given any two lines L, M that intersect at a unique point P in the plane, can you always draw a third line N through P such that the angle between N, L is a third of that between M, L?

  To give an explicit example: we can make a line that makes an angle of $\pi/3 = 60°$ with the origin by constructing a circle with radius 2 around the origin, drawing a line perpendicular to the x-axis through $(1, 0)$, finding their intersection P, and drawing the line through the origin and P.



  Can you draw a line that makes an angle of $\pi/9 = 20°$ with the origin? In other words, can you construct $\cos(20°)$?

- **Squaring the Circle**: Given a circle C with radius 1, can you construct a point P such that the distance from P to the origin is the same as the circumference of C? In other words, can you construct π?

We answer these questions in these notes. To do this, we will need the following three tools:

1. The concept of **dimension**, as covered in our earlier notes.
2. The following theorem on constructible numbers, that we proved in week 8:

   Theorem. Let a be any constructible value. Then there are constructible values b, c such that a is a root of the polynomial

   $$x^2 + bx + c.$$

3. The concept of a **field extension**.

This third concept is not one that we have discussed yet! We define it here:

1.1 Field extensions: definitions, examples.

Definition. (will be filled in when time allows!)