Math/CS 120: Intro. to Math	Professor: Padraic Bartlett

Homework 9: Equivalence Relations

Due Friday, Week 5 *UCSB 2014*

Hey! These problems are slightly harder than normal. Do **two** of the **five** problems below!

1. Consider the collection of all polynomials with real-valued coefficients, which we denote as $\mathbb{R}[x]$. Take any polynomial $h(x) \in \mathbb{R}[x]$.

   Consider the following relation:

   $$\equiv_h := \{(f(x), g(x)) \mid \exists q(x) \in \mathbb{R}[x], f(x) - g(x) = q(x)h(x)\}.$$

   For example, if $h(x) = x - 2$, we would say that $f(x) = x^2 - 4$ is equivalent to $g(x) = x^2 - 3x + 2$, because

   $$f(x) - g(x) = x^2 - 4 - (x^2 - 3x + 2) = 3x - 6 = 3(x - 2) = 3h(x).$$

   Prove that $\equiv_h$ is an equivalence relation on the collection of all polynomials.

2. Let's continue to work with the equivalence relation $\equiv_h$ defined in the problem above. Denote any equivalence class containing a polynomial $f(x)$ as $[f(x)]_h$: this represents the collection of all of the polynomials equivalent to $f(x)$ under $\equiv_h$.

   We can define $+, \cdot$ on these equivalence classes as follows: $[f(x)]_h + [g(x)]_h = [f(x) + g(x)]_h$, and $[f(x)]_h \cdot [g(x)]_h = [f(x) \cdot g(x)]_h$.

   Set $h(x) = x^2 + 1$ for this problem, and let $(\mathbb{R}[x]/h(x))$ denote the collection of equivalence classes for $\mathbb{R}[x]$ under h.

   (a) Show that $[x]_h \cdot [x]_h + [1]_h = [0]_h$.

   (b) Consider the map $\varphi : \mathbb{C} \to (\mathbb{R}[x]/h(x))$ defined by $h(a + ib) = [a + bx]_h$. Is φ a bijection? Does φ preserve addition and multiplication: in other words, for any $a, b, c, d \in \mathbb{R}$, do we have

   - $\varphi((a + ib) + (c + id)) = [a + bx]_h + [c + dx]_h$, and
   - $\varphi((a + ib) \cdot (c + id)) = [a + bx]_h \cdot [c + dx]_h$?

3. There are two possible equivalence relations on the two-element set $\{1, 2\}$:

   - The relation $\sim_A$ that says that $1 \sim_A 1$ and $2 \sim_A 2$ are true, and that $1 \sim_A 2$ and $2 \sim_A 1$ are false.
   - The relation $\sim_A$ that says that $1 \sim_A 1$, $2 \sim_A 1$, $1 \sim_A 2$, and $2 \sim_A 2$ are all true.

   For any n, let E_n denote all of the possible equivalence classes on the set $\{1, 2, \ldots n\}$. Find a recurrence relation for E_n. (For example: we defined the Fibonacci numbers recursively via the equation $f_{n+1} = f_n + f_{n-1}$. In this problem, you are being asked to create an recursive expression for E_n; i.e. a way to express E_n in terms of earlier terms.)

4. Let X be some set, and $S \subseteq X \times X$ be a relation on the set X.

   (a) Prove that there is some equivalence relation $R \subset X \times X$ such that $S \subseteq R$. (If you're not comfortable with the subset notation: this is a relation R with the property that if xSy, then xRy must also hold.)

   (b) Take the collection of all equivalence relations T such that $T \supseteq S$. Let $R_S = \bigcap_T T$ denote the intersection of all of these equivalence relations: in other words, R_S is a relation where xR_Sy holds only if xTy holds for each T.

   Show that R is an equivalence relation. (We think of R as the equivalence relation **generated** by S.)

   (c) Consider the set $X = \{1, 2, 3, 4, 5\}$ and relation $S = \{(1, 2), (3, 4)\}$: i.e. S thinks that $1S2, 3S4$ are both true, and that all other relations are false.

   Find the equivalence relation R_S generated by S.

5. (a) Give an equivalence relation $\sim$ on $\mathbb{R}$ such that the equivalence class of any element is size 4.

   (b) Give an equivalence relation $\sim$ on $\mathbb{R}$ such that there are infinitely many equivalence classes of $\mathbb{R}$ under $\sim$, and each equivalence class has the same cardinality as $\mathbb{R}$.