Math/CS 120: Intro. to Math	Professor: Padraic Bartlett

Homework 7: Set Theory

Due Friday, Week 4 — *UCSB 2014*

Do **three** of the **six** problems below!

1. In class, someone asked if it was possible for a set A to contain itself: in other words, if $A \in A$ is possible given our rules for sets!

   Surprisingly, it turns out that our existing axioms are not strong enough to stop this from happening. However, the idea of a set containing itself is kind of bothersome, so mathematicians decided to create the following axiom:

   Axiom. (Axiom of Foundation) Every nonempty set contains an element that is disjoint from the original set. In symbols:

   $$\forall A, ((A \neq \emptyset) \Rightarrow (\exists x(x \in A) \wedge (x \cap A = \emptyset)))$$

   Suppose you have this axiom, along with the other axioms we've created thus far (empty set, pairing, union, power set, comprehension, infinity.) Prove that it is impossible for $A \in A$ to hold for any set A.

2. A related impossible object is an infinite sequence of nested sets, defined as follows: we call A an **infinite sequence of descending sets** if for any $x \in A$, there is some other $y \in A$ such that $x \in y$. The idea is that if such a set existed, we could build an infinite[1] chain

   $$x_1 \ni x_2 \ni x_3 \ni x_4 \ni \ldots$$

   by repeatedly picking for each set x_i the next set x_{i+1} such that it's in A and contained by x_i.

   This seems awful, right? Show that such a set cannot exist, if you have the above axiom of foundation along with all of our other axioms.

3. We constructed the natural numbers as follows:

   - Take any inductive set S (one must exist, by the axiom of infinity.)
   - Using power set, form the collection of all subsets of this inductive set, $\mathcal{P}(S)$.
   - Using comprehension, form the subset T of $\mathcal{P}(S)$, consisting of all of the subsets of S that are inductive.
   - Again using comprehension, take the intersection of all of the elements of T.
   - Call this set $\mathbb{N}_S$.

[1] Side note: the LaTeX command for the backwards "$\in$," "$\ni$," is "\ni." This is adorable.

In class, we claimed that this set didn't really care about S, in the following sense: for any two inductive sets R, S, we proved that $\mathbb{N}_R = \mathbb{N}_S$. As part of this proof, we looked at the intersection $C = \mathbb{N}_R \cap \mathbb{N}_S$ of these two sets, and claimed that this intersection was inductive — however, we left the proof of this claim for the homework!

Prove this here: for any two inductive sets S, T, show that $C = \mathbb{N}_R \cap \mathbb{N}_S$ is an inductive set. (Hint: as with all problems involving definitions, show that C satisfies the definition of being an inductive set.)

4. Suppose that a, b are members of the natural numbers $\mathbb{N}$ as defined via sets in class thus far – that is, think of elements of $\mathbb{N}$ as sets, i.e. $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \ldots$. Also suppose that $b \in a$. Prove that a is not a subset of b: in symbols, $a \not\subseteq b$.

5. Assume that $\mathcal{P}(A) = \mathcal{P}(B)$. Prove that $A = B$.

6. Given any set A, we can always form by the union axiom the set

$$\bigcup A := \bigcup_{A' \in A} A'.$$

Suppose that A, B are elements of $\mathbb{N}$ such that $A = S(B)$, where S is the successor function $S(B) = B \cup \{B\}$. Prove[2] that the set $\bigcup A$ is equal to the set B.

(Hint: form the collection $X = \{B \in \mathbb{N} \mid \bigcup S(B) = B\}$, using our axioms. If you can show this is an inductive set, what can you conclude?)

[2] In this sense, $\bigcup$ "undoes" successor; if the successor function is like $+1$ on the natural numbers, then $\bigcup$ is like -1.