Homework 6: Set Theory

Due Friday, Week 3 | *UCSB 2014*

Do **one** of the **three** problems below!

1. A **singleton** is any set of the form $\{x\}$: that is, any set that contains exactly one element. Suppose that we could form the set
$$S = \{A \mid A \text{ is a singleton set.}\}.$$
Show that we can use this claimed set to create a contradiction[1].

2. The **generalized union axiom** is the following axiom:

   Axiom. (Axiom of Union, full version.) Given any set A, there is a set C whose elements are exactly the members of the members of A. In symbols:
$$\forall A\ \exists C\ [(x \in C) \Leftrightarrow (\exists A'(A' \in A) \wedge (x \in A'))].$$
We denote this set as
$$\bigcup_{A' \in A} A'.$$
In class, we gave instead the "weakened" version of this axiom:

   Axiom. (Axiom of Union, simple version.) Given any two sets A, B, we can make a set whose members are those sets belonging to either A or B (or both!) In symbols:
$$\forall A, B\ \exists C\ \forall x[x \in C \Leftrightarrow ((x \in A) \wedge (x \in B)].$$
We write this set C as $A \cup B$, for shorthand.

   Using literally only the axioms we've given in class other than the simple union axiom, show that the stronger union axiom can "prove" our weaker union axiom. (In other words: start with the axioms we've gone through in class - (simple union) + (generalized union). Use just these results to prove the simple axiom of union.)

3. (a) Find two sets a, B such that $a \in B$, but $\mathcal{P}(a) \notin \mathcal{P}(B)$.
   (b) Prove that if $a \in B$, we must have
$$\mathcal{P}(a) \in \mathcal{P}\left(\mathcal{P}\left(\bigcup_{b \in B} b\right)\right).$$

[1] This is a little similar to how in class we looked at Russell's paradox, linked here, which pointed out how the object $R = \{A \mid A \notin A\}$ creates a contradiction if it is a set.