Math/CS 120: Intro. to Math	Professor: Padraic Bartlett

Homework 5: More Cardinality

Due Friday, Week 3 *UCSB 2014*

Do **three** of the **six** problems below!

1. A real number r is called **algebraic** if and only if there is some degree n and integer constants $a_0, \ldots a_n$ such that r is a root[1] of the following polynomial:

$$a + 0 + a_1 x + \ldots + a_n x^n.$$

Most numbers you know are algebraic: for example, all of $\mathbb{Q}$ is (they're roots of the polynomial $qx - p$), as is $\sqrt{2}$ (it's a root of $x^2 - 2$).

Prove or disprove the following statement: $\mathbb{N}$ has the same cardinality (size) as $\mathcal{A}$, the collection of all algebraic numbers.

2. Define the **Cantor set** $\mathcal{C}$ as follows:

- Start with the interval $[0, 1]$. Call this set C_0.
- Remove the middle-third of this set, so that you have $[0, 1/3]$ and $[2/3, 1]$ left over. Call this set C_1.
- Remove the middle-third of those two sets, so that you have $[0, 1/9], [2/9, 1/3], [2/3, 7/9], [8/9, 1]$ left over. Call this set C_2.
- Repeat this process!

Define $\mathcal{C}$, the Cantor set, as the set made by taking all of the elements x such that x is in C_i, for every i.

(a) Find an element in $\mathcal{C}$.

(b) Show that $\mathcal{C}$ contains infinitely many elements.

(c) Can you make a bijection between $\mathcal{C}$ and $[0, 1]$?

3. Let X denote the set made out of all possible sequences of natural numbers with finite length: i.e. for every element x of X, there is some length k such that x looks like some string $(n_0, n_1, \ldots n_k)$, where the $n_1 \ldots n_k$'s are all natural numbers. Is this set the same cardinality as $\mathbb{N}$?

4. Let Y denote the set made out of all possible sequences of natural numbers of infinite length: i.e. for every element y of Y, y looks like some string $(n_0, n_1, \ldots)$, where the elements n_i are natural numbers. Is this set the same cardinality as $\mathbb{N}$?

[1] A **root** of a polynomial is a number you can plug into that polynomial and get 0. For example, 2 is a root of the polynomial $x^2 - 4$, because plugging in 2 for x yields 0.

5. We say that a set is **countably infinite** if there is a bijection from that set to $\mathbb{N}$. Suppose that $A_1, A_2, A_3, \ldots$ is a sequence of countably infinite sets. Define

$$B = A_1 \cup A_2 \cup A_3 \cup \ldots = \bigcup_{n=1}^{\infty} A_n.$$

Show that $|B| = |\mathbb{N}|$.

6. Given sets A, B, C and functions $f : A \to B, g : B \to C$, form the function $h = g \circ f : A \to C$. In other words, h is the function from A to C defined by $h(a) = g(f(a))$.

   For each of the following claims, provide a proof or a disproof:

   (a) If h is injective, then f is injective.

   (b) If h is injective, then g is injective.

   (c) If h is surjective, then f is surjective.

   (d) If h is surjective, then g is surjective.