Math/CS 120: Intro. to Math — Professor: Padraic Bartlett

Homework 17: Complex Numbers

Due Friday, Week 10 — *UCSB 2014*

Solve **three** of the following **six** problems. As always, prove your claims/have fun!

1. Let z_1, z_2, z_3 be three elements of $\mathbb{C}$ such that

   - $z_1 + z_2 + z_3 = 0$, and
   - $|z_1| = |z_2| = |z_3|$.

   Show that these three points, as graphed in $\mathbb{C}$, form the vertices of an equilateral triangle.

2. A **n^{th}-root of unity** is a complex number $\omega \in \mathbb{C}$ such that $\omega^n = 1$. For example, -1 and 1 are the only two 2^{nd}-roots of unity, because they are the only two complex numbers ω such that $\omega^2 = 1$.

   (a) Find all of the fifth roots of unity. (Hint: use polar coördinates. In other words, think of elements of $\mathbb{C}$ as numbers of the form $re^{i\theta}$.)

   (b) Let ω be a n^{th}-root of unity not equal to 1. Show that

   $$\sum_{k=0}^{n-1} \omega^k = 0.$$

   (Hint: Draw all of these powers of ω on the complex plane for some small values of n, like 3 or 4. Graphically, why is this sum 0?)

3. Take all of the n-th roots of unity $\omega_1, \ldots \omega_n$. Find a closed form for the product

   $$\prod_{k=1}^{n} \omega_k.$$

   (It should be very simple.)

4. Consider the following game of "Lights-Out:"

   - We start with a circle of $n \geq 2$ lights $b_1, \ldots b_n$, some of which are on and some of which are off.
   - The only available moves are the following: suppose that there is some divisor d of n, and some bulb b_i such that for any $k \equiv i \mod d$, b_i has the same state as b_k. Then we can switch the states of all of these lights at once.

   Suppose your game starts with exactly one light on. Can you turn all of the other bulbs on?

5. One of the best reasons for working in $\mathbb{C}$ is that every polynomial of degree n has n roots! We don't have time to prove in this class, so we leave it here for the HW:

   Theorem. (The Fundamental Theorem of Algebra:) Every complex polynomial $p(z)$ with degree n has n (possibly repeated) roots in the complex plane. In other words, we can factor every degree n polynomial into n complex roots: i.e. we can always find constants such that

   $$p(z) = C \cdot \prod_{k=1}^{n} (z - r_i).$$

6. As it turns out, there is a far stronger analogue to the above theorem, which says (basically) that we can factor not just polynomials, but entire power series into their roots! This theorem is incredibly difficult to prove (you could easily spend a pair of quarters on complex analysis and not get to it,) so we state it without proof below:

   Theorem. Weierstrass Factorization Theorem: every complex power series $f(x) = \sum a_n z^n$ can be written in the form

   $$e^{g(z)} x^n \cdot \prod_{\text{all roots } r_k \text{ of f}} \left(1 - \frac{z}{r_k}\right),$$

   for some natural number n and some other complex power series $g(x)$.

   Basically, this says that we can separate any complex power series into its roots (the $(1 - \frac{z}{r_i})$ - parts and the z^k part), times some $e^{g(z)}$-part that's never 0.

   If you apply this theorem to $\sin(z)$, you'll get the following formula:

   $$\sin(z) = z \cdot \prod_{n \neq 0, n \in \mathbb{Z}} \left(1 - \frac{z}{\pi n}\right).$$

   Also do not try to prove this.

   Instead, do the following: **use** the $\sin(z)$ identity above to prove the following result:

   $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$