Math/CS 120: Intro. to Math — Professor: Padraic Bartlett

Homework 13: More Limits

Due Friday, Week 7 — *UCSB 2014*

Do **one** of the following **three** problems!

1. Does the recursively-defined sequence

$$a_1 = 1,$$
$$a_{n+1} = \sqrt{1 + a_n^2}$$

converge? Prove your claim, **without** referring to or creating a closed form for a_n.

2. Suppose that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that the limits

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k^2\right), \lim_{n\to\infty}\left(\sum_{k=1}^{n} b_k^2\right),$$

both exist and are finite. Prove that the limit

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}(a_n \cdot b_n)\right)$$

also exists and is finite.

3. For any positive integer k, the k-**hailstone** sequence $\{h_n\}_{n=0}^{\infty}$ is defined as follows:

   - Define $h_0 = k$.
   - If h_n is odd, define $h_{n+1} = 3h_n + 1$.
   - If h_n is even, define $h_{n+1} = \frac{h_n}{2}$.

   For example, the following sequence is the 13-hailstone sequence:

$$13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots$$

   The **Collatz conjecture**, an open problem in mathematics, is the claim that no matter what number you start with, this sequence will eventually reach 1.

   (a) Write a computer program to verify that the Collatz conjecture is true for all positive natural numbers less than 1000. Attach your code!

   (b) In our example sequence where we started at 13, we got to 1 after 9 steps, i.e. at the 9th entry of our sequence. Using your program, determine which number less than 1000 takes the most steps to get to 1. How many steps does it take? (Again, attach your code or other proof.)

   (c) (Open/extra credit/hard.) Show that every hailstone sequence contains a 1.