Math/CS 120: Intro. to Math	Professor: Padraic Bartlett

Homework 11: Introduction to Analysis

Due Friday, Week 6 — *UCSB 2014*

This problem set relies on several concepts we talked about in class! We list them here:

- **Limits**: We say that a sequence $\{a_n\}_{n=0}^{\infty}$ has the limit L as n approaches infinity if "the terms of $\{a_n\}_{n=0}^{\infty}$ go to L as n gets large." Formally, we say

$$\lim_{n\to\infty} a_n = L \Leftrightarrow \forall \epsilon > 0, \exists N \forall n > N, |a_n - L| < \epsilon.$$

  Essentially, for any amount of "close to L" we want, there is some N past which if we pick a_n's with $n > N$, we have that a_n is as close to L as we wanted.

- **Cauchy**: We say that a sequence $\{a_n\}_{n=0}^{\infty}$ is Cauchy if "the terms of $\{a_n\}_{n=0}^{\infty}$ get close to each other as n gets large." Formally, we say

$$\{a_n\}_{n=0}^{\infty} \text{ is Cauchy} \Leftrightarrow \forall \epsilon > 0, \exists N \forall m, n > N, |a_n - a_m| < \epsilon.$$

  Essentially, for any amount of "close to L" we want, there is some N past which if we pick a_m, a_n's with $m, n > N$, we have that a_n is as close to a_m as we wanted. It's *like* having a limit, except that you don't actually know where the sequence is going; just that the terms are getting close to each other!

- **Measure**: We say that any function μ that takes subsets of $\mathbb{R}$ to $\mathbb{R}^{\geq 0}$ is a **measure**[1] if it satisfies the following properties:

  1. **Base cases**: $\mu(\emptyset) = 0$, $\mu([0, 1]) = 1$.
  2. **Translation/Scaling**: For any set A that μ is defined on and any $x \in \mathbb{R}, \lambda \in \mathbb{R}^+$, we have $\mu(A) = \mu(A + x)$ and $\mu(\lambda \cdot A) = \lambda \cdot \mu(A)$.
  3. **Subadditive**: If $A \subseteq B$ are sets that μ is defined on, then $\mu(A) \leq \mu(B)$.
  4. **Additive**: If $\{A_i\}_{i=1}^{\infty}$ is a collection of disjoint sets that μ is defined on, then $\sum_{i=1}^{\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{n} A_i\right)$.

Here are the problems! Do **three** of the following **six**.

1. Consider the following sequence:

$$a_1 = 1, a_n = a_{n-1} + \frac{1}{n^2}.$$

   Show that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

2. When we defined the real numbers using Cauchy sequences in class, many people asked why we couldn't just use the definition of "sequences with limits" in place of the stranger "Cauchy" definition. This problem explains why we used Cauchy! Specifically, consider the following "alternate" construction for the real numbers:

[1] Strictly speaking, a Lebesgue measure.

- Let S be the set of all sequences $\{a_n\}_{n=1}^{\infty}$ of rational numbers with the following property: there is some $L \in \mathbb{Q}$ such that $\lim_{n\to\infty} a_n = L$.
- Call two sequences with limits **equivalent** if they have the same limit: formally, say that $\{x_n\}_{n=1}^{\infty} \sim \{y_n\}_{n=1}^{\infty}$ if and only if there is some $L \in \mathbb{Q}$ such that $L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.
- Define the real numbers as $S/\sim$, i.e. the collection of all equivalence classes of sequences of rational numbers under $\sim$.

Explain why this construction does not give you $\mathbb{R}$.

3. Take our rules for being a measure, and remove the "translation/scaling" property: call such an object a "meamaybe." Find a meamaybe that assigns 0 or 1 to every subset of $\mathbb{R}$, and explain why it satisfies the three properties of being a meamaybe.

4. A set $S \subseteq \mathbb{R}$ has an **upper bound** if there is some element $U \in \mathbb{R}$ such that

$$\forall s \in S,\ U \geq s.$$

Similarly, a set $S \subseteq \mathbb{R}$ has a **lower bound** if there is some element $F \in \mathbb{R}$ such that

$$\forall s \in S,\ F \leq s.$$

Show that if a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a limit $L \in \mathbb{R}$, then it has an upper and lower bound.

5. Show that if a sequence $\{a_n\}_{n=1}^{\infty}$ has a limit, then it is Cauchy.

6. For any interval $[a, b]$, we define the process of removing the "middle-n-th" as simply removing the open interval

$$\left(\frac{b+a}{2} - \frac{b-a}{2n}, \frac{b+a}{2} + \frac{b-a}{2n}\right).$$

from this set. For example, removing the middle-3-rd from $[0, 1]$ is just removing

$$\left(\frac{1}{2} - \frac{1}{6}, \frac{1}{2} + \frac{1}{6}\right) = \left(\frac{1}{3}, \frac{2}{3}\right)$$

from $[0, 1]$, which leaves us with the set $[0, 1/3] \cup [2/3, 1]$. Note that if we started with a closed interval, this always leaves us with two closed intervals. The Cantor set C_∞ was created iteratively, by setting $C_0 = [0, 1]$ and saying that if C_n was defined, we can make C_{n+1} by removing the middle-third from each interval in C_n. From there, we defined C_∞ as the intersection of all of these sets.

Consider the following "Cantor-like" set S_n:

- $S_0 = [0, 1]$.
- If we have constructed S_{n-1}, create S_n out of it by removing the middle $\frac{1}{4^n}$-th from each interval in S_{n-1}.

Set S_∞ as the intersection of all of these sets. What is the measure of S_n?