Latin Squares: Transversals and counting of Latin squares
Jenny Zhang, Yian Huang
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Author: Jenny Zhang

First, let's preview what mutually orthogonal Latin squares are. Two Latin squares $L_1 = [a_{ij}]$ and $L_2 = [b_{ij}]$ on symbols $\{1, 2, ...n\}$, are said to be orthogonal if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}), $1 \leq i \leq n$, $1 \leq j \leq n$.

Now, let me introduce a related concept which is called transversal. A transversal of a Latin square is a set of n distinct entries such that no two entries share the same row, column or symbol.

This is an example of mutually orthogonal latin squares.

L_1=

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

L_2=

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

We have (2,2), (3,1), (1,3), (1,1), (2,3), (3,2), (3,3), (1,2), (2,1)

Theorem 1. A Latin square has an orthogonal mate if and only if it can be decomposed into n disjoint transversal.

If we consider the exactly n cells of the Latin square L_2 all of which contain the same fixed entry h say$(1 \leq h \leq n)$, then the entries in the corresponding cells of the Latin square L_1 must all be different, otherwise the squares would not be orthogonal. Since the symbol h occurs exactly once in each row and column of the latin square L_2, we see that the n entries of L_1 corresponding to the entry h in L_2 is a transversal.

Orthogonal latin squares exist for all orders $n \notin \{2, 6\}$. For n=6, there is no pair of orthogonal squares, but we get close. We have an example which contain 4 disjoint transversals.

$$\begin{bmatrix} 1_a & 2 & 3_b & 4_c & 5 & 6_d \\ 2_c & 1_d & 6 & 5_b & 4_a & 3 \\ 3 & 4_b & 1 & 2_d & 6_c & 5_a \\ 4 & 6_a & 5_c & 1 & 3_d & 2_b \\ 5_d & 3_c & 2_a & 6 & 1_b & 4 \\ 6_b & 5 & 4_d & 3_a & 2 & 1_c \end{bmatrix}$$

Partial Latin Square

Partial latin square of order n is a matrix of order n in which each cell is either blank or contains one of $\{1, 2, ..., n\}$, and which has the property that no symbol occurs twice within any row or column. A cell which is not blank is said to be filled. A partial latin square with every cell filled is call latin square. The set of partial latin squares of order n is denotes by $PLS(n)$, and the set of latin squares of order n by $LS(n)$. We say $P_1 \in PLS(n)$ is said to be completable if there is some $L \in LS(n)$ such that L contains P. On the other hand, P is said to be maximal if the only partial latin square which contains P is P itself.

We coin the name k-plex of order n for a $K \in PLS(n)$ in which each row and column of K contains exactly k filled cells and each symbol occurs exactly k times in K. The entries on a transversal of a latin square form a 1-plex.

We say that two plexes in the same square is parallel if they have no filled cells in common. The union of an a-plex and a parallel b-plex of a latin square L is an (a+b)-plex of L. However it is not in general possible to split an (a + b)-plex into an a-plex and a parallel b-plex.

Next, let me present a few theorems about the k-plex:

Theorem 1. If $n > 2$ then there exists $L \in LS(n)$ which contains a k-plex for each k satisfying $0 \leq k \leq n$.

Proof: If $n > 2$ and $n \neq 6$, a celebrated result says that there are two orthogonal latin squares of order n. So there will be transversal in the latin square, then there will at least be 1-plex in the latin square. For n = 6 there is no pair of orthogonal squares, however we found a latin square order 6 which contains 4 parallel transversals that I talked about before. What we just proved is that if 0 k n and n ¿ 2 then there is a completable k-plex of order n. However, our next result shows that not all k-plexes are completable.

Theorem 2. If $1 < k < n$ and $k > \frac{n}{4}$ there exists an uncompletable k-plex of order n.

Theorem 3. For $k \leq \frac{n}{4}$ every k-plex of order n is contained in a $(k+1) - plex$ of order $4n$.

Author:Yianhuang

1 Introduction

In this presentation, I'm going to introduce the concept of Latin Rectangle and the counting of $2 \times n$ latin rectangle.

2 latin rectangle $L = l_{ij}$

For $1 \leq k \leq n$, k x n latin rectangle is the $k \times n$ array $L = l_{ij}$ with entries from $\{1, 2, ..., n\}$ such that the entries in each row and each column are distinct.
For k=2, a 2 x n latin rectangle is the $2 \times n$ array $L = l_{ij}$ with entries from $\{1, 2, ..., n\}$ such that the entries in each row and each column are distinct.
We can also say a $k \times n$ latin rectangle part of $n \times n$ latin square.

3 reduced latin rectangle $R_i j$

We say $R_i j$ is reduced latin rectangle if The first row is $(1, 2, ...n)$ and the first column is $(1, 2, ...n)^T$. For instance, this is considered a reduced latin rectangle.

1	2	3	4
2	4	1	3
3	1	4	2

For Reduced Latin Rectangle, we have some very interesting things to talk about. But today I'm going to discuss a reduced Latin Rectangle in a $2 \times n$ form, which is like:

1	2	3	4
2	4	1	3

The cool thing of a reduced latin rectangle is once we find all possible latin rectangle to a reduced latin rectangle, we can us Permutation to find all possible latin rectangles! For instance, for a 2x3 latin rectangle we can easily know there are basically 2 types of reduced latin rectangles which are:

1	2	3
2	3	1

and

1	2	3
3	1	2

And if written as tuples, there are six permutations of the set $\{1, 2, 3\}$, namely: (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1), which means we can substitute with any of its 6 permutations all permutations have 2 types of reduced latin rectangles. In this case, if we substitute $\{1, 2, 3\}$ with $\{1, 3, 2\}$ we get new latin rectangles!

1	3	2
3	2	1

and

1	3	2
3	1	2

And since we have 6 of such permutations, we can say we have $6 \times 2 = 12$ 3×2 Latin Rectangles in total. In general, for a $2 \times n$ Latin Rectangle, since we have $n!$ permutations, the possible Latin rectangle L_ij is $n!R_ij$

4 number of $2 \times n$ reduced latin rectangles

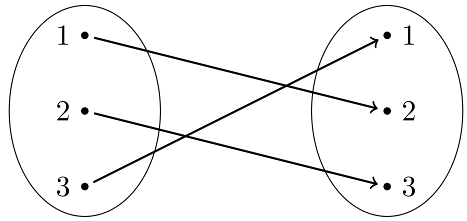
For $2 \times n$ latin rectangles, since we've already know once we find the total number of R_ij, we can find the total number of latin rectangle L_ij.

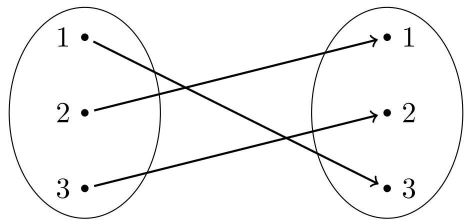
To count R_ij, our easy way is by using bijection:

In below 2×3 case, we can easily "draw" all the possibilities and we can eventually find 2 graphs.

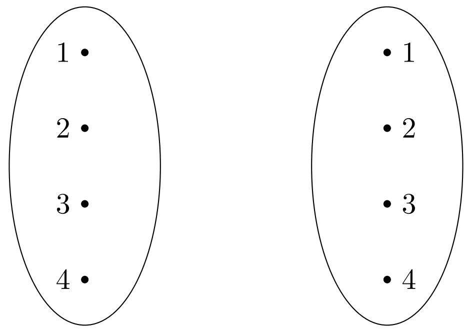
In below 2×4 case, after we randomly link 2 vertexes together, we find a pattern in which is very similar to the 2×3 case but for each bijection due to the 1 number difference, we can draw 3 kind of graphs, so the whole number of latin rectangle after we link 2 vertexes together is 3 and since we can randomly like give 3 kinds of first random vertexes link, all the possibilities and we can eventually find is $3 \times 3 = 9$ graphs.

when we come to 2×5 case, after we randomly link 2 vertexes together, like what we just did to 4, we find a pattern in which is very similar to the 2×4 case but for each bijection due to the 1 number difference, we can draw 4 this kind of of graphs, and in each of the





graph, when we continue to randomly link 2 vertexes together, we find 2 kinds of graphs, one kind is a 2×3 latin rectangle and the other kind is 3 latin rectangle that appears in the 2×4 case. Then we notice a interesting pattern: which is the number of 2×5 latin rectangle is (n-1) times number of 2×3 and 2×4 latin rectangle. This is all what this class is going to talk about. The next class I' m going to discuss how to solve this equation: $N_n = (n-1)(N_{n-1} + N_{n-2})$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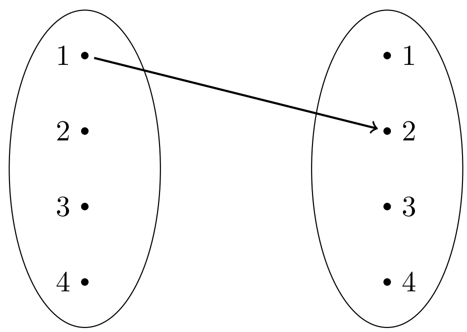
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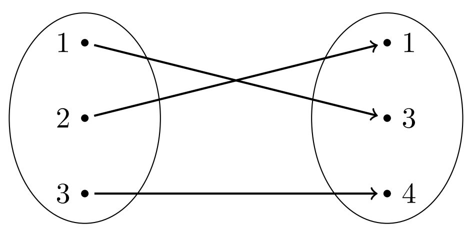
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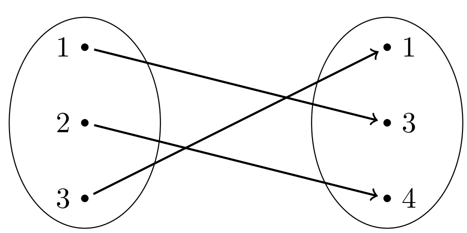
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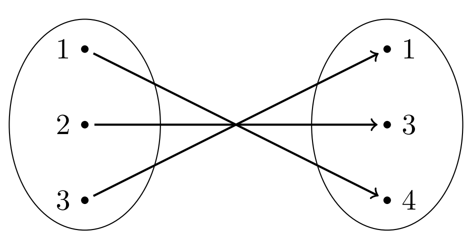
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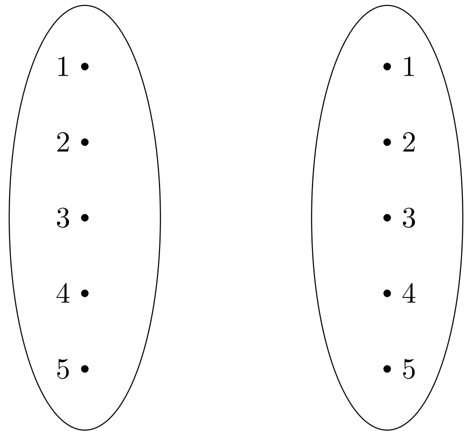
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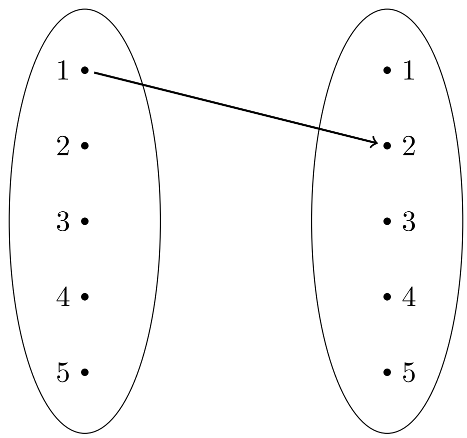
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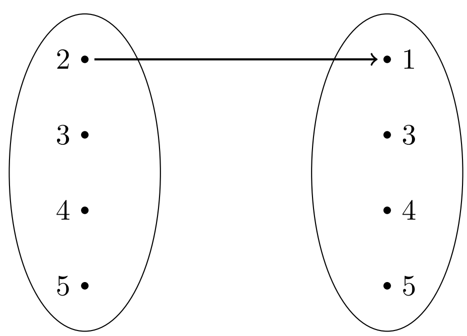
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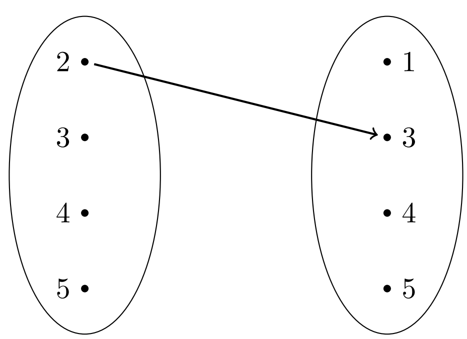
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