Crossing Game

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There are many games that can be played in regards to knot theory. In the following document, we will begin to explore one such game.

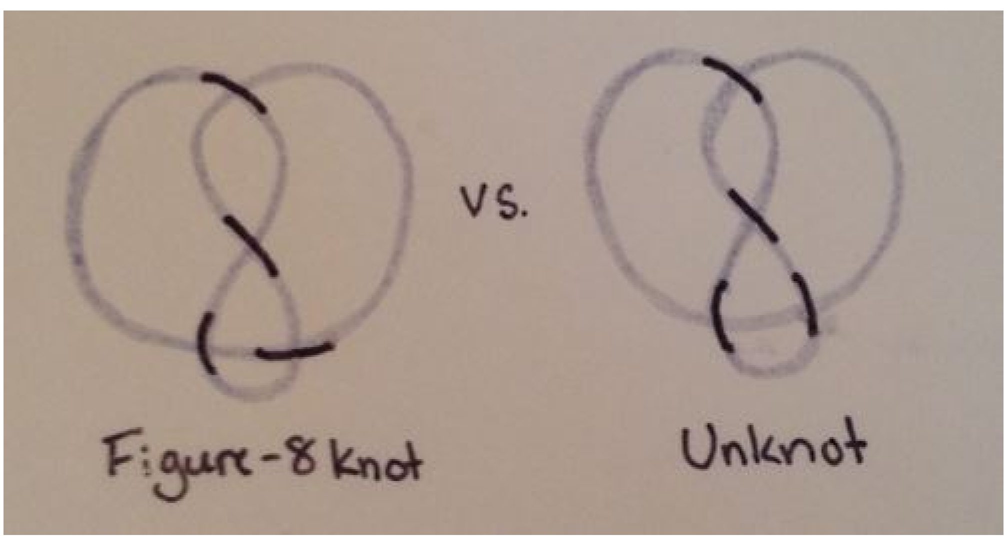
1 Crossing Changes

Definition. *A **crossing change** is when a strand is switched from going over another strand to going under that strand at a crossing, or vice versa.*

Theorem. *We can always obtain a regular projection of the unknot from a diagram of a knot from crossing changes.*

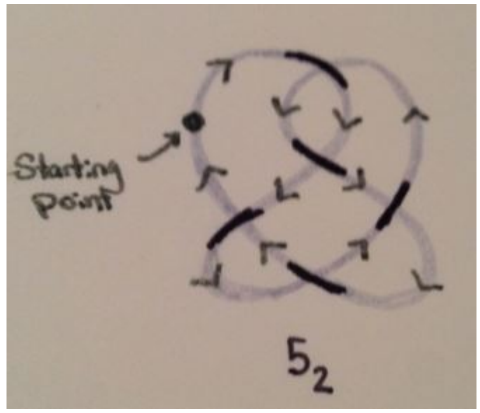
This is easily seen if you take a regular projection of the knot and, as you trace along it, change crossings such that the strand that you are following always goes over the other strand at a crossing the first time you come to it (and under the second time). This means the string is essentially piled on top of itself and, when looking from the side, we may observe that the knot does not have any crossings and therefore is the unknot.

Observe in the following diagram that one crossing switch can change the figure-8 knot to the unknot:



Definition. *A knot K is **alternating** if the crossings alternate between over and under as one moves along the regular projection of K.*

The following is an example of an alternating knot (this is most easily seen if you begin at the starting point and follow the arrows along the knot):



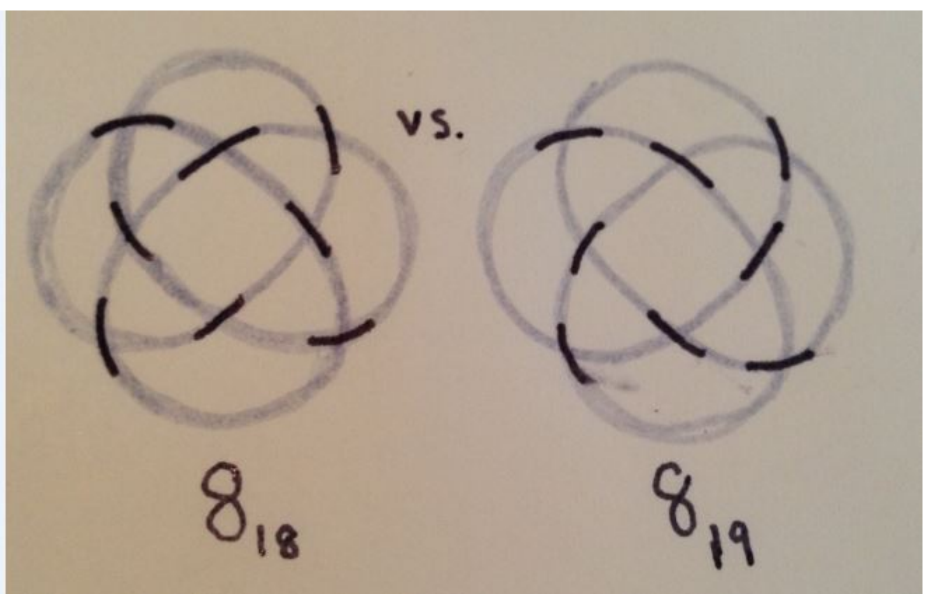
Theorem. *Any alternating knot of three or more crossings is not equivalent to the unknot.*

Proof of this theorem is omitted because it is beyond the scope of this class. The full proof is included in reference (4).

All of the knot diagrams that we have seen so far have been of alternating knots. This might lead one to believe that all knots that are represented with **minimal crossings** are alternating.

Definition. *A **minimal crossing number** of a knot is the smallest possible number of crossings in any projection of the knot. The minimal crossing number is a knot invariant.*

Surprisingly, some knots that are represented with minimal crossings are non-alternating. For example consider the following knots (knot 8_{18} and knot 8_{19}) with the same number of minimal crossings:

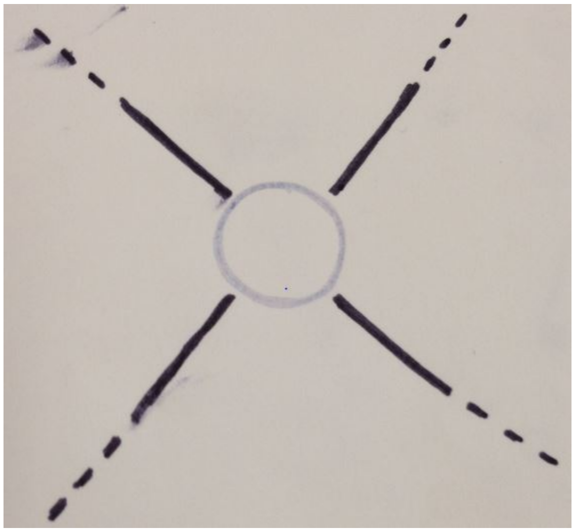


We note that knot 8_{18} is alternating while knot 8_{19} is not.

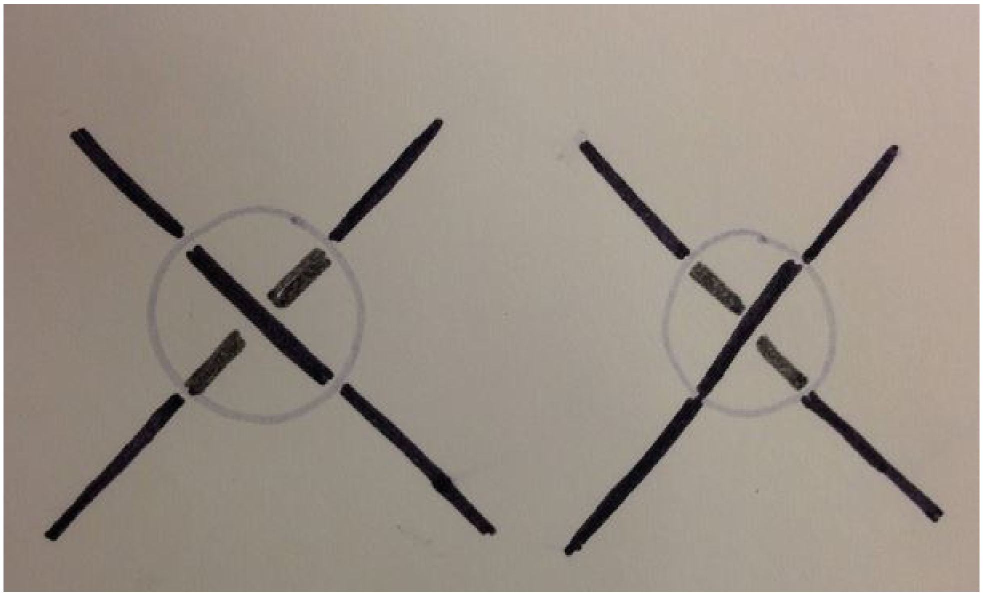
2 Crossing Game

The layout of the crossing game entails a regular projection of a knot that is missing its crossing information. This is a two-player game where one player is labeled "K" (for knot) and the other player is labeled "U" (for unknot). Players in this game take turns choosing crossing information, meaning that they pick a crossing with missing information and decide which strand goes over and which strand goes under. The objective of Player U is to create a projection of a knot that is equivalent to the unknot, while the objective of Player K is to create a projection of a knot that is not equivalent to the unknot.

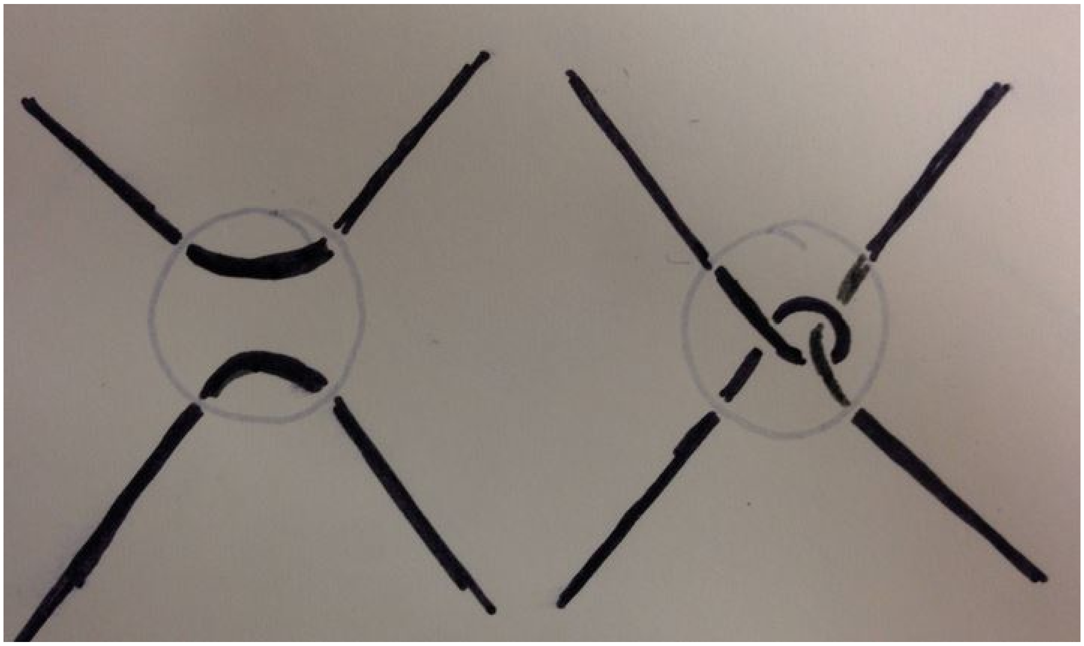
The following is how we denote a crossing that is missing information:



The following diagram depicts the allowed moves in this game:

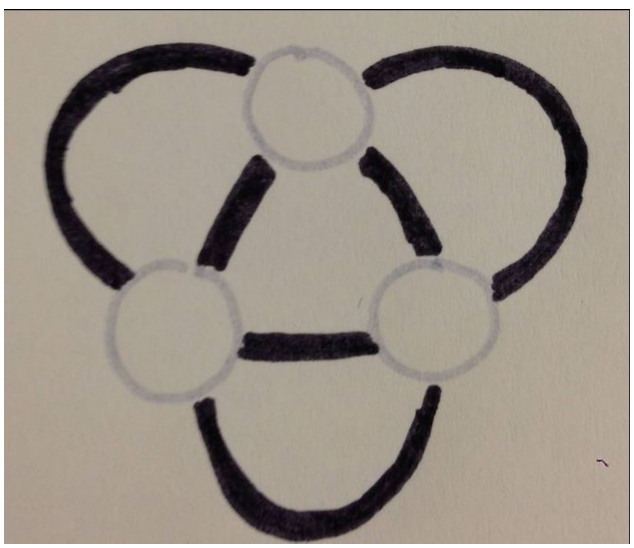


The following diagram depicts moves that are not allowed in this game:



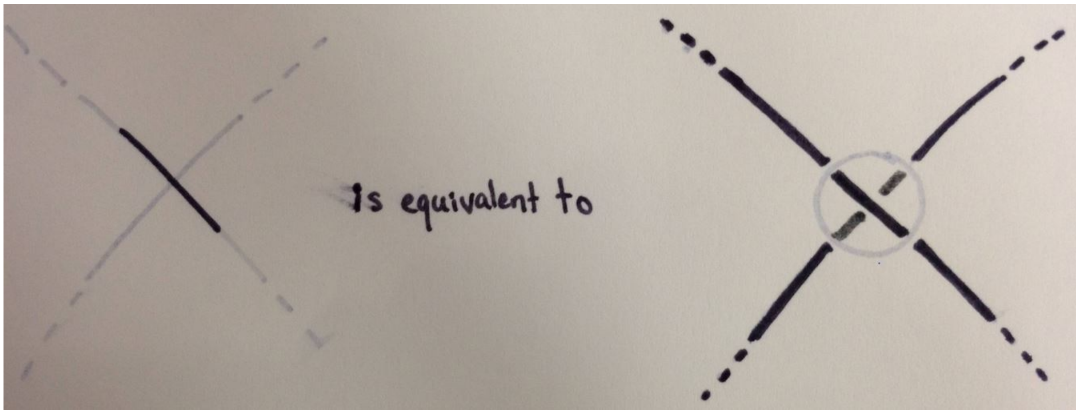
Next, we have an example of this game with the most basic playable knot, the trefoil.

We begin our game with a diagram that looks like the following:

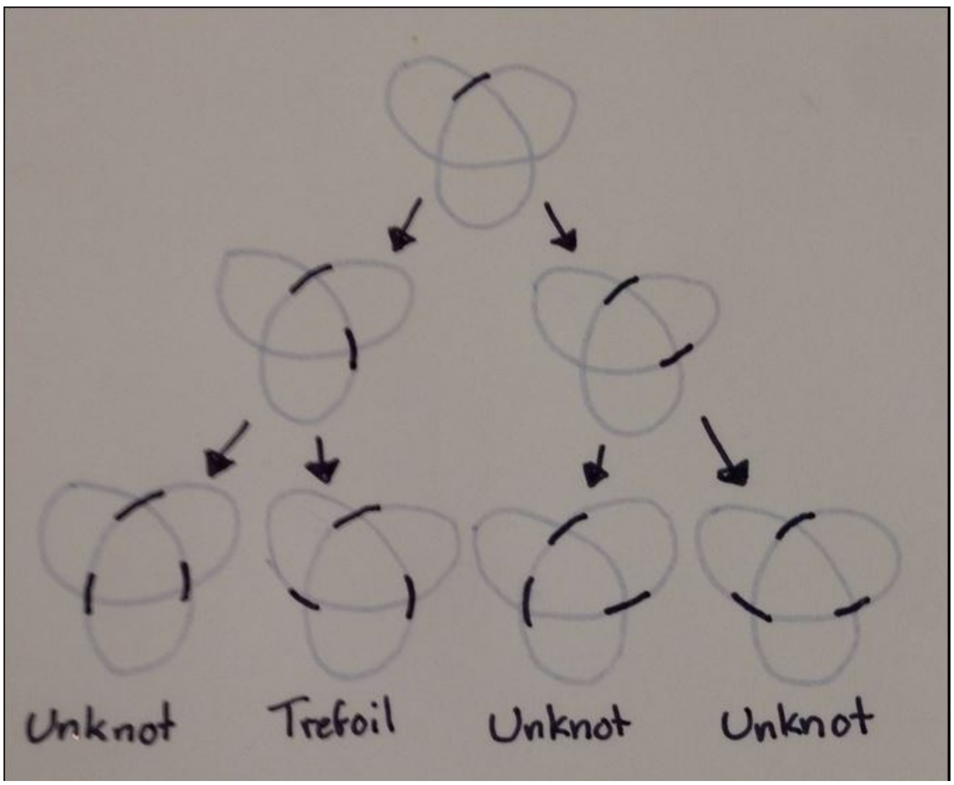


Note that the first move in this game is arbitrary.

Note: for the purpose of this write-up, it is often easier to see the crossings when they are denoted slightly differently. This gives us the following:



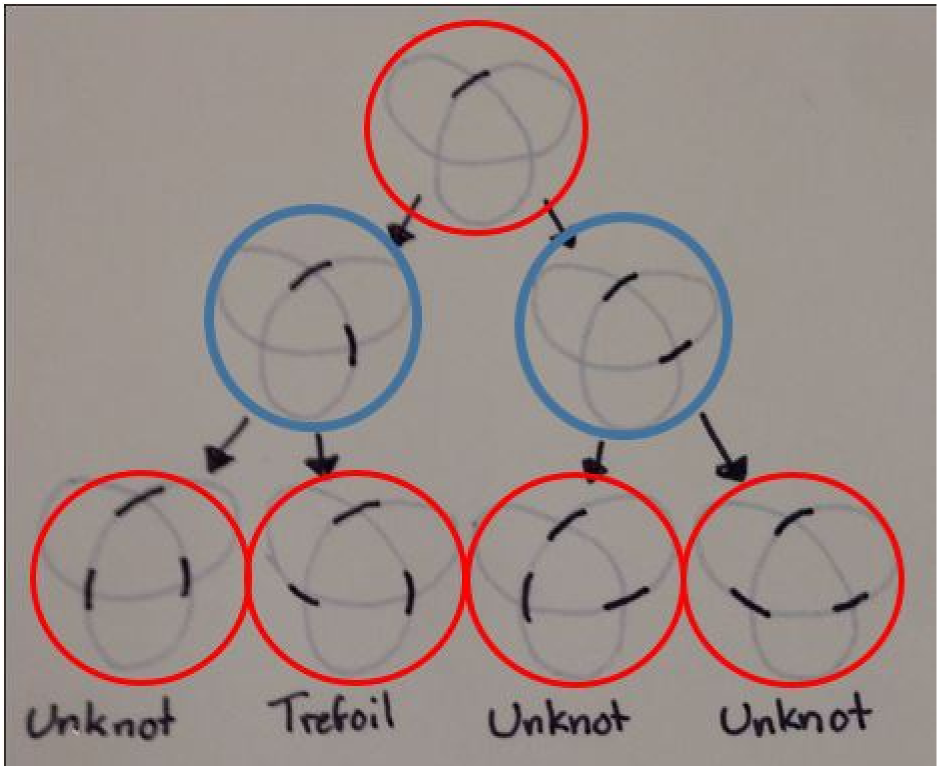
Observe the following diagram of the possible outcomes of the crossing game played with this "game board".



A question we might want to ask about this game is, "Who has a winning strategy?" We must consider the following two cases:

- Player K goes first.

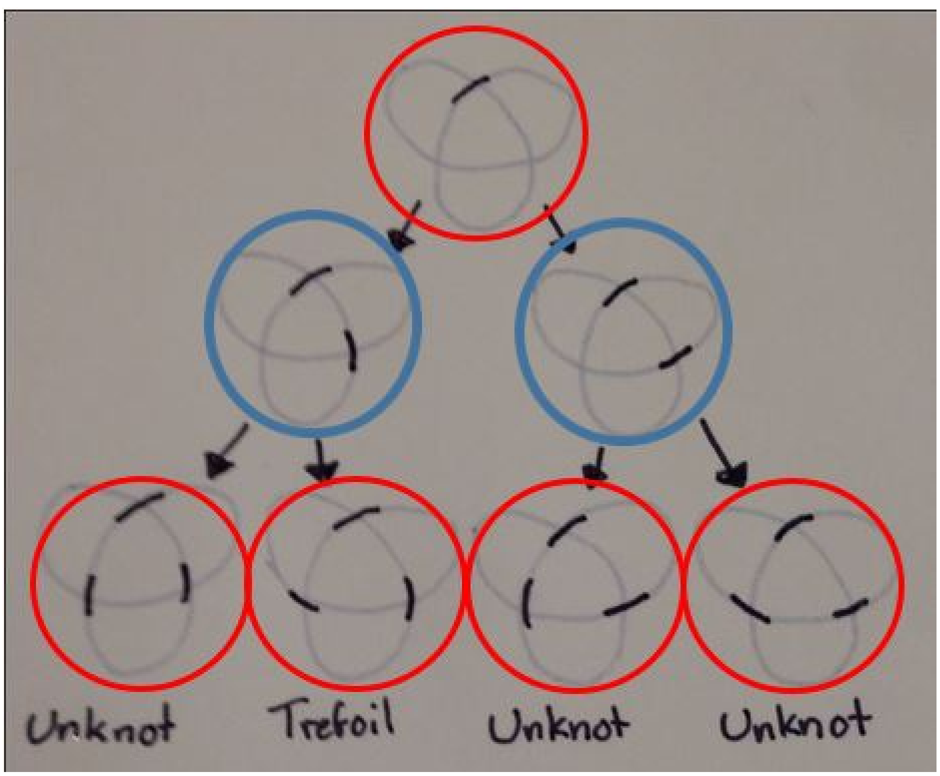
  Player K's possible moves are the ones circled in red and Player U's possible moves are the ones circled in blue.



Player U makes the second move. If Player U makes the move that is leftmost in the diagram above, Player U forces Player K to choose the unknot in the next turn, thereby always allowing Player U to win.

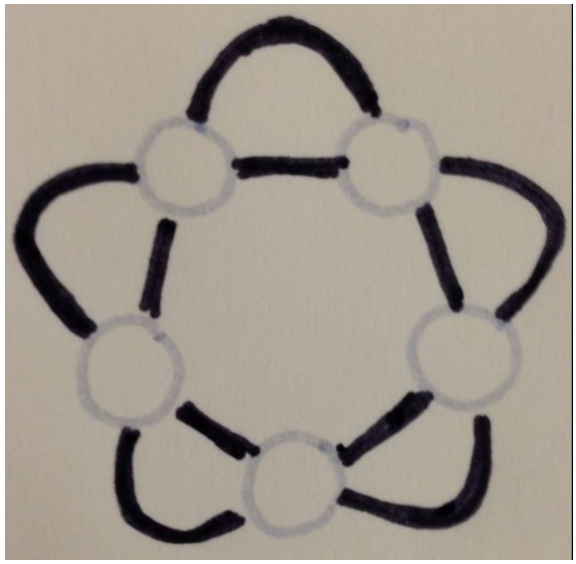
- Player U goes first.

  Player K's possible moves are the ones circled in red and Player U's possible moves are the ones circled in blue.



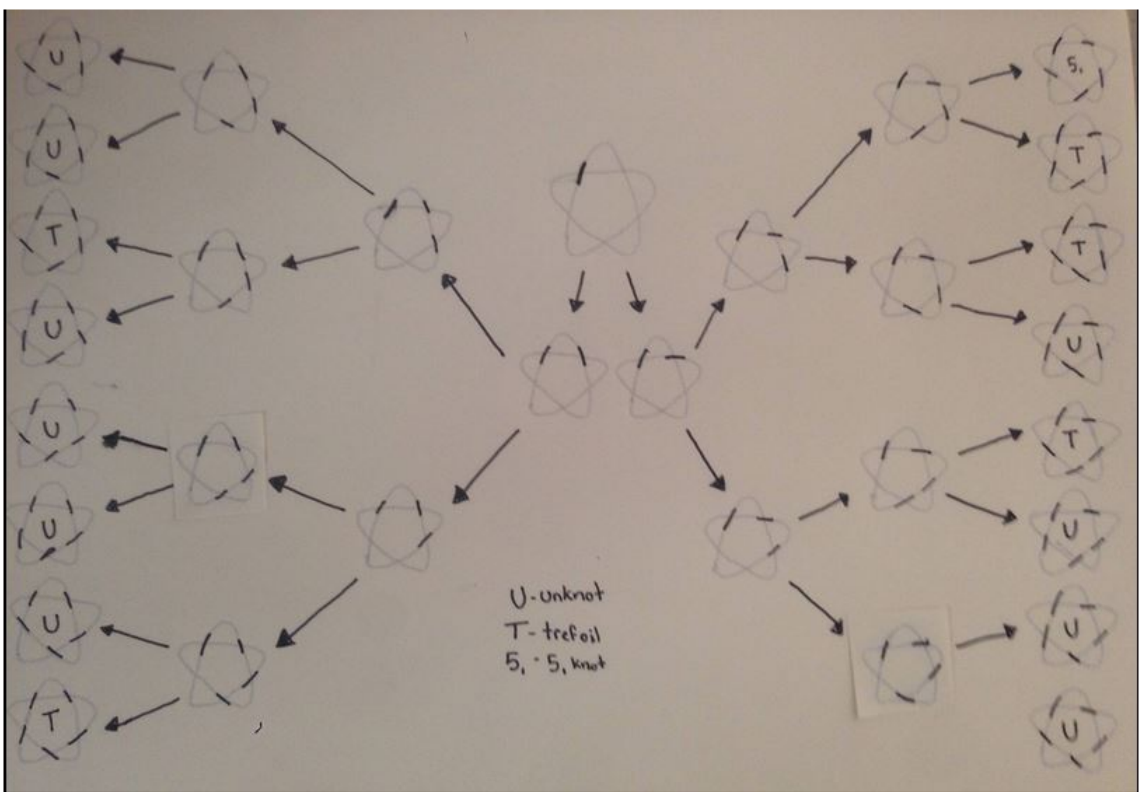
Player U makes the last move. Notice that no matter what move Player K makes, Player U is always able to choose the unknot on the last move, therefore always allowing Player U to win.

This example of the game is fairly simple, so a more interesting example follows.
Consider the following game board:



In order to allow the reader to more easily see the strategy in this game, we will restrict the game to moves being made strictly clockwise in regard to where the next move occurs.

The following diagram contains all of the possible moves for the players:

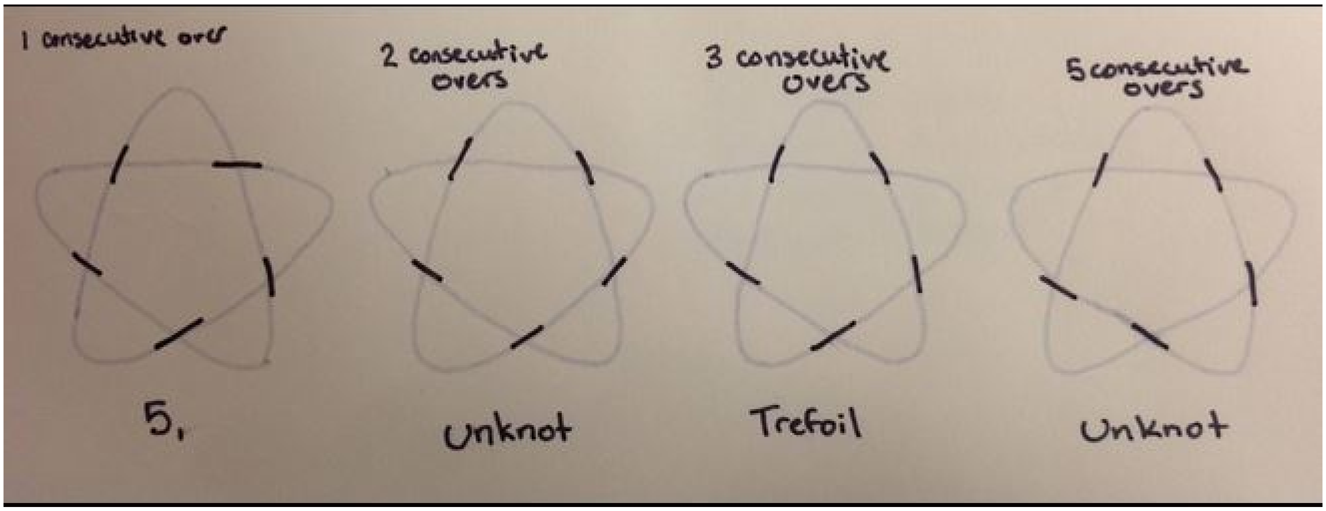


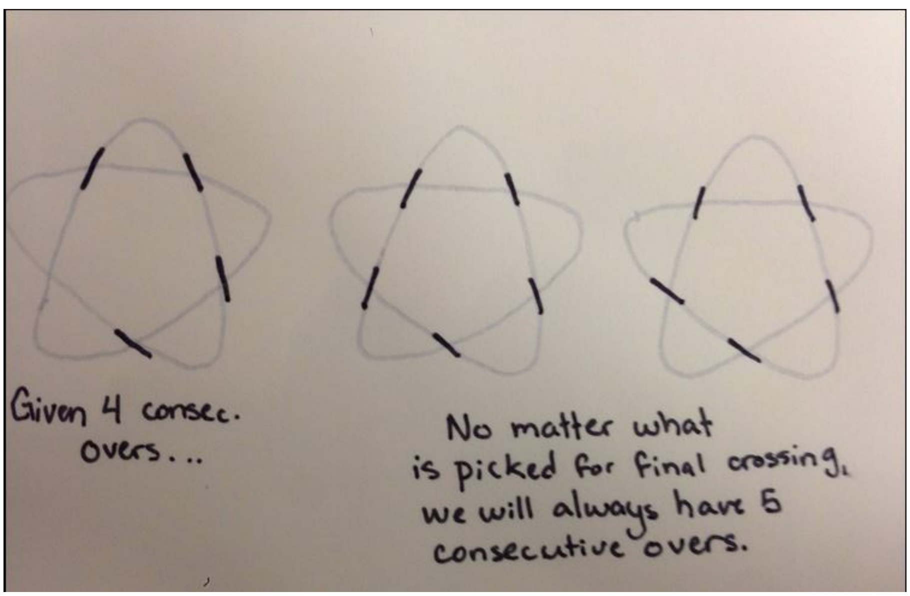
Given a "completed game" (one with all of the crossings filled in) of this board, how do we know who wins?

It turns out that there is a simple way to check who won at the end of this game.

Definition. *The number of* ***consecutive overs*** *is, when following along a strand of a regular projection of a knot, the largest number of times that that strand goes over another strand consecutively.*

We can figure out who wins this game by looking at the number of consecutive overs.



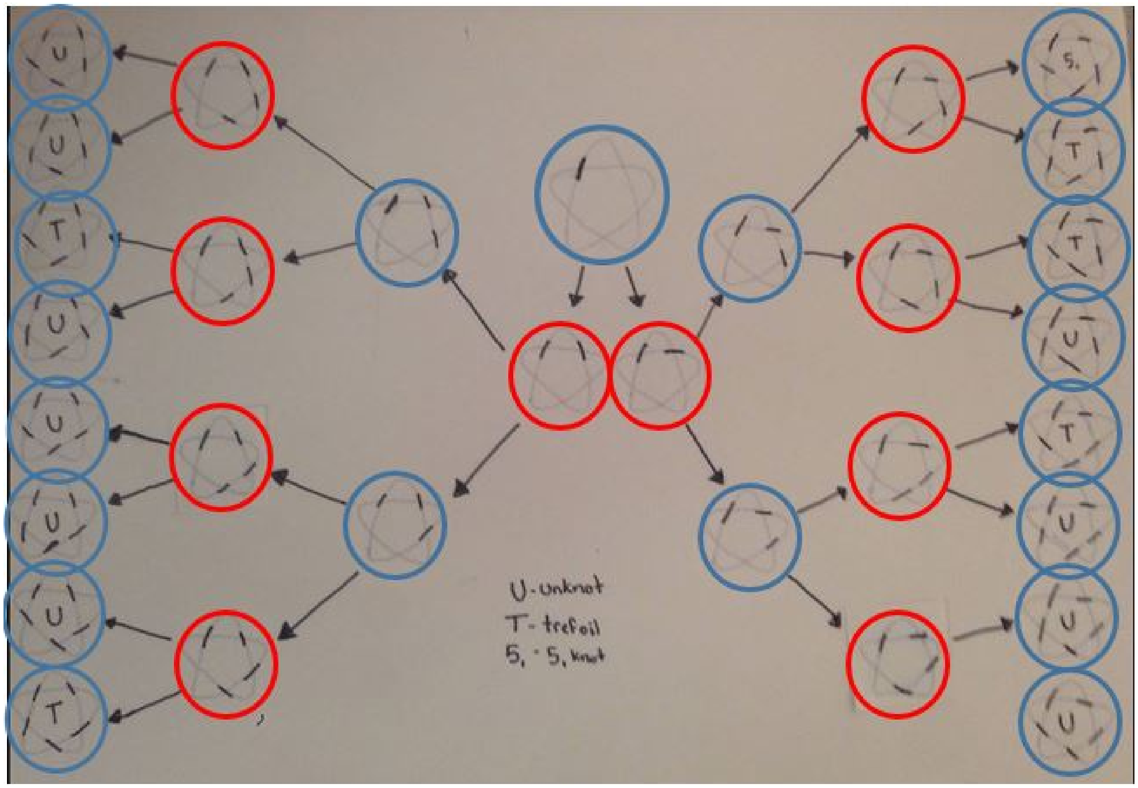


Note: the first move is again arbitrary.

We again have two possible cases that we must consider for this game:

- Player K goes first.

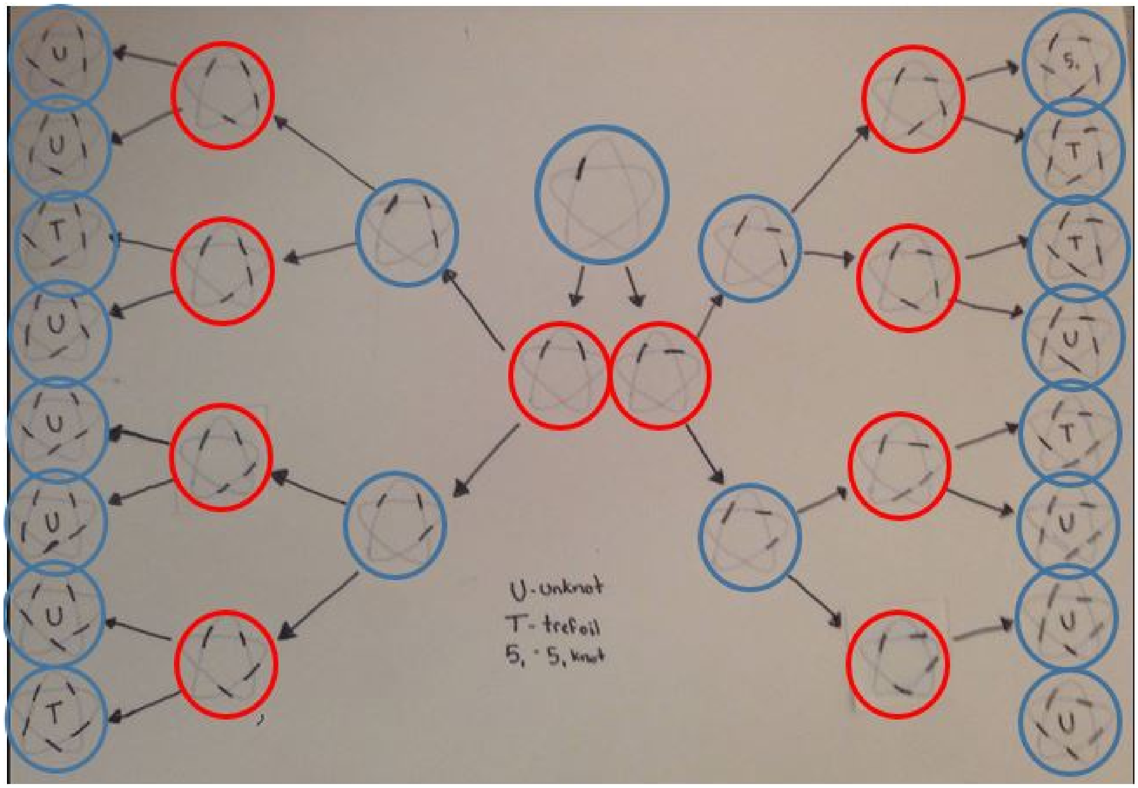
  Player K's possible moves are the ones circled in blue and Player U's possible moves are the ones circled in red.



If, at Player U's first move the leftmost option is chosen, then no matter what is chosen on Player K's second move, Player U's second move can always force Player K to choose between the unknot and the unknot on the final turn, thereby allowing Player U to win.

- Player U goes first.

  Player K's possible moves are the ones circled in blue and Player U's possible moves are the ones circled in red.



In this case, Player U makes the final move. The only time when the unknot is not an option on the final move is the two options at the top right of the diagram. Therefore, as long as, on Player U's second turn, Player U does not choose the top right move (which is not forced), then Player U always has the unknot as an option in the final move, therefore giving Player U a winning strategy.

3 References

1. *http://www.csuchico.edu/math/mattman/NSF/Lecturenotes.pdf*

2. *http://www.math.washington.edu/~mathcircle/mathhour/talks_2014/henrich-slides.pdf*

3. *http://arxiv.org/pdf/1003.4494v1.pdf*

4. *http://journals.cambridge.org/download.php?file=%2F569_5FB3C9A390AD930DCE46D2D626108903_journals_PSP__PSP109_03_S0305004100069887a.pdf&cover=Y&code=fbb39dcc992a5803ff41ae708a1ef5e1*

5. The Knot Book by Colin C. Adams