Link Smoothing Game Theory Continued

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In the last presentation, we introduced the following concepts:

- N-position, P-position, K-position, and L-position link shadows
- Using a link diagram's corresponding black and white graphs to simplify the link smoothing game into the contraction and deletion of edges
- n-edge blowouts
- We listed a few graph properties that indicate an L-position graph

Next, we will introduce more properties of the link smoothing game being played on one of link diagram's respective planar graphs.

1 Lemma 1

Suppose a game is played on link shadow diagram D corresponding to connected planar graph G. L's final move results in an L win iff L is allowed to play on a cut edge or a loop.

Proof. By definition, a connected link diagram may only be disconnected by a smoothing at a crossing if a nugatory crossing is present. A nugatory crossing corresponds to a loop in one graph associated to the link diagram and a cut edge in the dual graph. □

2 Lemma 2

Any independent pair of moves in a link smoothing game commutes. To be more precise, suppose we are playing a game on embedded planar graph G corresponding to link shadow D and let e and e' be distinct edges in G. Then if play P (i.e. edge deletion or contraction) is performed on edge e followed by play P' on edge e', the resulting game is indistinguishable from the game produced by performing play P' on edge e' followed by playing P on e.

Proof. It is clear that the graph resulting from a sequence of edge deletions and contractions is not dependent upon the order in which edges were deleted or contracted. (i.e. the end result is invariant of order of contractions/deletions).

□

Next, we will describes classes of diagrams with the properties of being N and P position games.

3 Theorem 1

Suppose shadow diagram D corresponds to a graph G. Let G be a graph that can be constructed from a single vertex by a sequence of 2-edge blowouts. Then K has a winning strategy in a game on D iff K plays second. Thus, D is a P-position game.

Proof. We observe that if we have a 2-edge blowout from a vertex, we can undo this by deleting an edge, and then contract the other one, or vice versa.

With this in mind, we claim that if we have a graph constructed from a single vertex, by a sequence of 2-edge blowouts, then K has a winning strategy if K goes first.

As a reminder, K wins if we end up with any knot including the unknot, which in the graph representation, is just a single vertex.

We proceed with the backward direction of this proof by induction.

Base Case: We start with a graph constructed from one 2-edge blowout from a vertex. We see that if player L deletes an edge, then player K can just contract the other one, in which case, the graph becomes a single vertex, and player K wins. If player L contracts an edge, then player K can just delete the other one, in which case, we end up with a single vertex, which means player K wins.

Inductive Step: We assume that if we have a graph constructed from n 2-edge blowouts from a vertex, K has a winning strategy if K goes second.

We need to show that K has a winning strategy if we have a graph constructed from $n+1$ 2-edge blowouts from a vertex. We can see this is indeed the case, in that if L starts by contracting an edge, then K can just delete the corresponding edge. If L starts by deleting an edge, then K can just contract the other one. In both cases, we get back to the configuration of a n 2-blowouts graph from a vertex, in which L goes first. By assumption, this is a configuration that K has a winning strategy for if K goes first. Thus, we have shown that K has a winning strategy on a graph constructed from 2-edge blowouts from a vertex.

To show the forward direction, it is enough to show that if K goes first, then K does not have a winning strategy. If K starts off by deleting an edge, the L can delete the corresponding edge, which results in the disconnection of the graph, which in the shadow diagram, translates to the configuration being separated. If K starts by contracting an edge, then L can contract the corresponding edge, which translates to disconnecting the shadow diagram.

Thus, we see that if K goes first, then K does not have a winning strategy, which is equivalent to saying if K has a winning strategy, then K cannot go first.

□

4 Corollary 1

Suppose link shadow diagram D corresponds to graph G. Let G be a graph that can be constructed from a single vertex by a sequence of 2-edge blowouts followed by a single 1-edge blowout. Then K has a winning strategy iff K plays first. Thus, D is an N-position game.

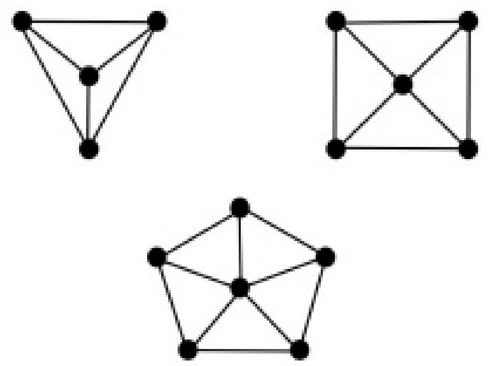
Proof. If K goes first, K can contract the 1-edge blowout. In doing so, K reduces the graph into the P-position game that was mentioned previously.

To prove the other direction, it is enough to show that if L goes first, L has a winning strategy. In this case, if L goes first, L can delete the 1-edge blowout, which translates to the graph being disconnected in the link shadow diagram.

□

There are examples where K has a winning strategy, despite the fact that the associated graphs are not of the 2- blowout form discussed in Theorem 1 or Corollary 1. Several such examples are pictured in Figure 1.

Figure 1



5 Proposition 1

1. If a graph G can be constructed from a single vertex by a sequence of 2-edge blowouts, then the following relationship between the number of vertices (v) and the number of edges (e) in G holds.
$$e = 2(v-1)$$

2. If a graph G can be constructed from a single vertex by a sequence of 2-edge blowouts along with a single 1-edge blowout, then the following relationship holds.
$$e - 1 = 2(v-2)$$

Proof. Observe that performing an n-edge blowout on a graph adds a single vertex as well as n edges. Suppose we obtain a graph G by a sequence of 2-edge blowouts on a single vertex. Then the graph must have $n+1$ vertices and $2n$ edges. If we were to perform a 1-edge blowout on the result, we would add one vertex and one edge, resulting in $n+2$ vertices and $2n+1$ edges. Together, these relationships yield the desired result. □

It is a natural question to ask if graphs not satisfying these equalities must be L-position. We answer this question with the following theorem.

6 Theorem 2

Let G be a connected, planar graph associated to a link shadow D. Let v and e denote the number of vertices and edges, respectively, in G. If e is even and $2(v-1) \neq 6 = e$, then D is an L-position diagram.

Proof. Since e is even, we may assume L moves first on D, else L has the last move and wins.

Next, the proof will be separated into two different cases.

1. $$e > 2(v-1)$$

   Suppose it is the case that $e > 2(v-1)$. We prove L has a winning strategy on D by induction on v. The assumptions above applied to the base case $v = 1$ gives rise to a graph G that is a bouquet of $e \geq 2$ loops. In L's first move, she can contract a loop, thus disconnecting the dual graph and winning. Let G be a connected planar graph with $v > 1$ vertices and an even number of edges, e, such that $e > 2(v-1)$. Suppose L has a winning strategy on all such graphs with fewer than v vertices. In L's first move she can contract any edge of G, thus resulting in a connected planar graph with $v-1$ vertices and $e-1$ edges. In K's next move she can either contract or delete an edge resulting in $e-2$ edges and either $v-2$ or $v-1$ vertices respectively. Presumably, K's move will not disconnect the diagram, thus it suffices to prove $e-2 > 2(v-2-1)$ and $e-2 > 2(v-1-1)$. The result, then, follows by induction since these two inequalities follow readily from the assumption that $e > 2(v-1)$.

2.

$$e < 2(v-1)$$

Next suppose $e < 2(v-1)$. We will prove that the dual of G, denoted G^* with v^* vertices and e^* edges, satisfies $e^* > 2(v^*-1)$. Proving this inequality holds will imply L has a winning strategy on the shadow associated to G^* by the argument given above, and thus, by duality, L has a winning strategy on D. (Recall that if one of G or G^* can be disconnected, L wins.) The planar graph G, embedded on the sphere gives rise to a polygonal decomposition satisfying $v-e+f=2$, where f is the number of polygon faces. By definition of the dual graph $e^*=e$, $v^*=f=2+e-v$. Thus, $e<2(v-1)$ implies $e^* < 2(2+e^*-v^*-1)$, which implies $e^* > 2(v^*-1)$.

□

From this theorem, we derive an immediate corollary for graphs with an odd number of edges.

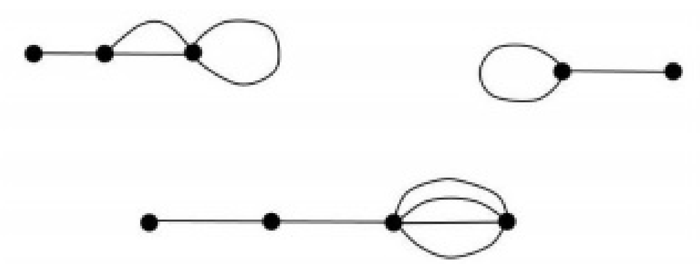
7 Corollary 2

Let G be a connected, planar graph associated to a link shadow D. Let v and e denote the number of vertices and edges, respectively, in G. If e is odd and $e > 2v-1$ or $e < 2(v-1)-1$, then D is an L-position shadow.

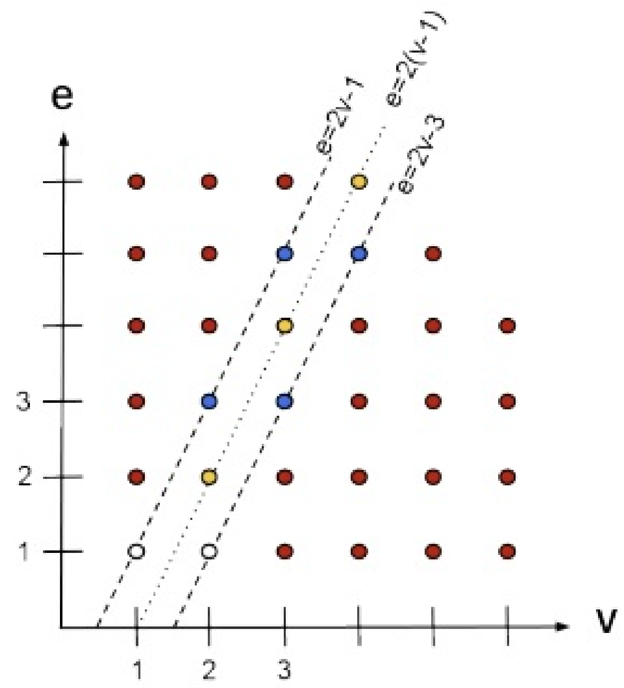
Proof. First, we note that if L moves first, L wins, so we assume that K moves first. It is easy to verify that no move performed by K will result in a diagram such that $e = 2(v-1)$. Thus, after K's move, we must be in the situation of Theorem 2. □

Figure 2 includes some examples of L-position graphs that satisfy $2(v-1)=e$.

Figure 2



8 Figure 3



The game class of a link shadow with connected graph G having v vertices and e edges is determined by the color of the lattice point (v,e). Red indicates an L-position graph, blue indicates the existence of both N-position and L-position graphs but no P-position graphs, and yellow indicates the existence of both P-position and L-position graphs but no N-position graphs. The pairs (2, 1) and (1, 1) are N- position.

References:

- *http : //arxiv.org/pdf/1109.4103v1.pdf*