Knot Theory Week 2: Tricolorability

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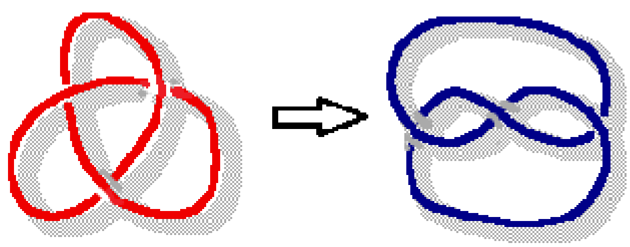
A central problem in knot theory is concerned with telling different knots apart. We introduce the notion of what it means for two knots to be "the same" or "different," and how we may distinguish one kind of knot from another.

1 Knot Equivalence

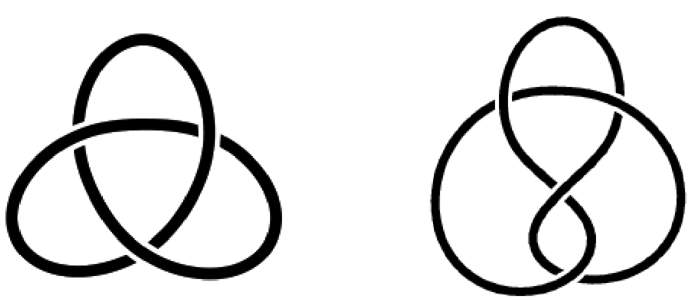
Definition. *Two knots are **equivalent** if one can be transformed into another by stretching or moving it around without tearing it or having it intersect itself.*

Below is an example of two equivalent knots with different regular projections (Can you see why?).



Two knots are equivalent if and only if the regular projection of one knot can be transformed into that of the other knot through a finite sequence of Reidemeister moves.

Then, how do we know if two knots are different? For example, how can we tell that the trefoil knot and the figure-eight knot are actually not the same?



An equivalent statement to the biconditional above would be: two knots are not equivalent if and only if there is no finite sequence of Reidemeister moves that can be used to transform one into another. Since there is an infinite number of possible sequences of Reidemeister moves, we certainly cannot try all of them. We need a different method to prove that two knots are not equivalent: a knot invariant.

2 Knot Invariant

A knot invariant is a function that assigns a quantity or a mathematical expression to each knot, which is preserved under knot equivalence. In other words, if two knots are equivalent, then they must be assigned the same quantity or expression. However, the converse is not true; if two knots are assigned the same invariant, it does not necessarily mean that they are equivalent. Different knots may have the same knot invariant, but depending on how "good" the particular invariant is, we may end up with fewer nonequivalent knots that are assigned the same value.

What we *do* know is that if two knots are assigned different values, then it must imply that they are nonequivalent knots.

A trivial example of a knot invariant is the constant invariant. We assign a constant a to all knots. This is a valid invariant, since "knots K_1 and K_2 are equivalent" implies "K_1 and K_2 have the same invariant, namely, a." However, this cannot provide any information on whether any two knots are different, since there is only one possible category, the one which every knot belongs to.

Other known knot invariants include Tricolorability, Crossing number, Bridge number, Unknotting number, Linking number and Polynomials. However, we do not yet know whether there exists a perfect knot invariant which can tell apart all knots from each other, or even all knots from just the unknot.

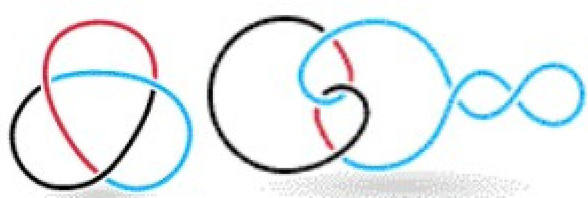
In this write-up, we focus on tricolorability.

3 Tricolorability

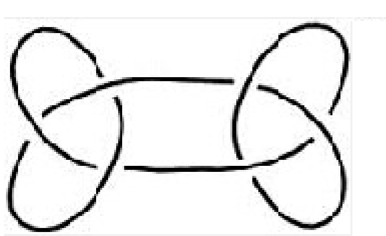
A **strand** in a knot diagram is a continuous piece that goes from one undercrossing to the next. The number of strands is the same as the number of crossings.
A knot is **tricolorable** if each strand of the knot diagram can be colored one of three colors, subject to the following rules:

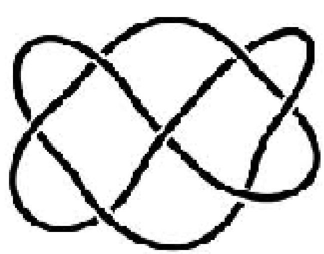
1. **At least two colors must be used; and**

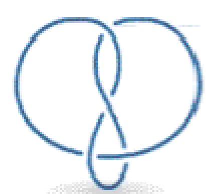
2. **At each crossing, the three incident strands are either all the same color or all different colors.**



As shown above, the trefoil is tricolorable. Now try the three knots below; are they tricolorable?

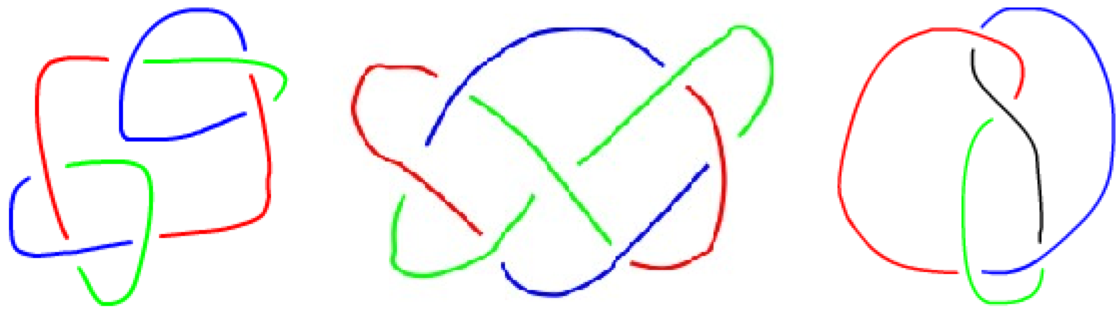






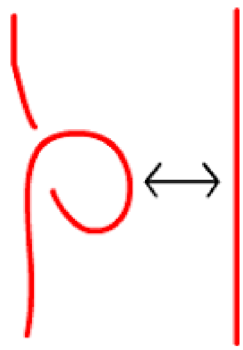
The left knot, known as the **Granny Knot**, is a composite knot obtained by taking the connected sum of two identical trefoil knots; The middle knot is 7_4 **Knot**. The right one is **figure-eight knot**.

The Granny Knot and the 7_4 Knot are tricolorable, but the figure-eight knot is not tricolorable. If we tried, we would run into a contradiction of the tricolorability rules, as the strand represented in black cannot be assigned any of red, green, or blue.

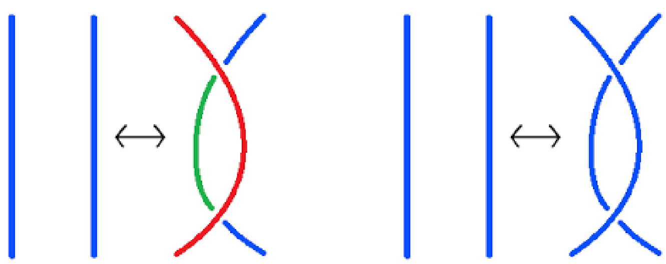


Remark. *Tricolorability is an invariant under Reidemeister moves.*

We show this by examining each of the three Reidemeister moves.

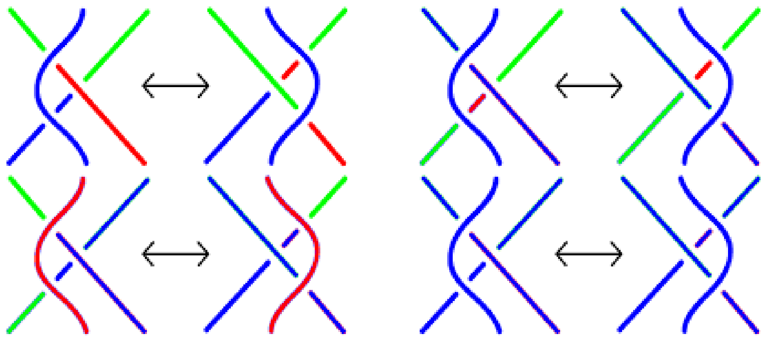


Move I, which allows us to untwist a knot, or to remove one crossing, preserves tricolorability as shown above. The entire twist is assigned a single color (since only two strands are actually used to represent a twist, we cannot use three distinct colors; therefore, they must have been the same color). After Move I, we can still keep the same color for the resulting strand. Conversely, if we had a strand and wanted to introduce a crossing, then we could color all strands that meet at the crossing to be the same color, and tricolorability will be preserved.



With Move II, first suppose that we want to introduce two new crossings onto two separate strands that have different colors assigned. Then, we could color each of the resulting three strands differently by introducing the third color, as shown in the image above. Conversely, if we were to pull apart two strands and thus remove two crossings, then we could use the two colors that were used before and remove the third color.

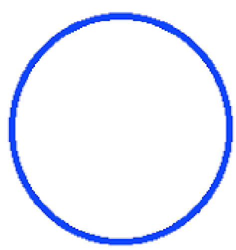
If the two strands were of the same color, then we could assign the same color to the third strand that results from introducing two crossings. Conversely, if we were to pull apart two strands of the same color that formed two crossings, we could keep the same color as well, and tricolorability is preserved.



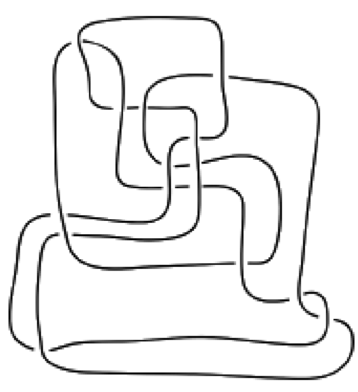
Now, if we have three strands, then sliding the top strand from one side of the crossing to another using Move III does not alter tricolorability, as shown in the images above. There are four possible cases: three cases in which one of the three crossings have strands with the same color, and one case in which all of the crossings have strands with the same color.

Now that we have shown that tricolorability is indeed a knot invariant, we have a useful fact: if we cannot color one regular projection of a knot using the tricolorability rules, then we also cannot do so with other regular projections of the same knot.

Remark. *The unknot is not tricolorable.*



The simplest regular projection of the unknot, a closed loop with no crossings, is not tricolorable, since there is only a single strand and we cannot use at least two distinct colors.

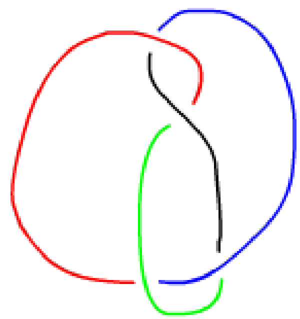


Therefore, we need not wonder whether this regular projection is tricolorable or not, if we already know that this is the unknot, since tricolorability is a knot invariant. Either every regular projection of a knot must be tricolorable, or every projection of a knot must be non-tricolorable.

Remark. *If a knot is tricolorable, then it is not equivalent to the unknot.*

Since the unknot is non-tricolorable, for any knot to be equivalent to the unknot, it must be non-tricolorable as well. Thus, a tricolorable knot is not equivalent to the unknot.

However, if a knot is non-tricolorable, then do we know that it is equivalent to the unknot? We illustrate the answer with the following example:



We have previously shown that this is a non-tricolorable knot. However, it is also nontrivial (i.e. not equivalent to the unknot). Thus, knowing that two knots are both non-tricolorable (or, both tricolorable) is not enough to actually tell whether those two knots are distinct. This is because tricolorability can only provide a binary classification: the knots that are tricolorable, and the knots that are non-tricolorable.

However, we can find the answer to our earlier question of proving that the trefoil is not equivalent to the figure-eight knot. Since the trefoil is tricolorable and the figure-eight knot is not, we know that they must be distinct.

Image sources:
Tricolorability, *Wikipedia*

http://www.math.cuhk.edu.hk/publect/lecture4/trefoil.html

http://www.math.utah.edu/mathcircle/notes/knots.pdf Knots: a handout for mathcircles