Knot Theory,
Problem Solving
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1 Introduction

Previously, we have introduced a game involving players to fill in crossings information, in order to achieve a knot or an unknot. We will now introduce a similar game, but instead of players trying to obtain knots and unknots, the objective will now be to obtain a link/unlink or a knot/unknot.

2 Link Smoothing Game

2.1 Basics

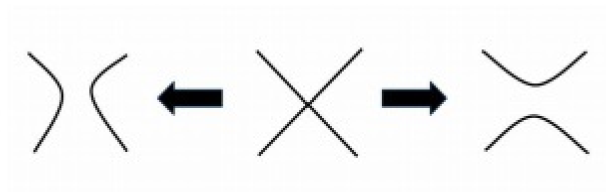
We begin by defining what a link and unlink is. A **link** is a collection of knots that do not intersect. However, they can be linked together, where one goes through the inside of the other. Like an unknot to a knot, the **unlink** is a specific type of link in which the two (or more) circles that make up the link are NOT linked together.

2.2 Players

In this game, there are two players. One of them is Link, while the other one is Knot. The goal for Link is to separate the original configuration, such that it is no longer connected in one piece. The goal for Knot is to keep the configuration in one piece. Note, the unknot is a knot, so Knot wins if an unknot is formed from all the moves. Similarly, Link wins if an unlink is formed. Call the two players player L and player K.

2.3 Moves

The moves of both players are similar to the moves in the Knot Crossing Game. Instead of over and under strands, we have horizontal and vertical smoothings. Below is a diagram of such smoothings.

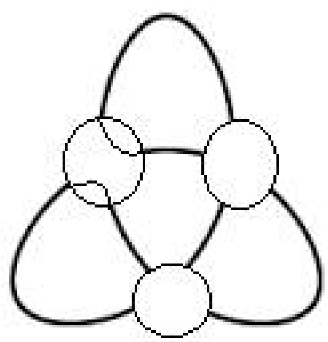


3 Strategies

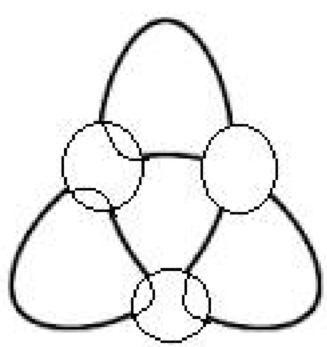
3.1 Trefoil

We begin by illustrating winning strategies of this game with the trefoil knot with crossings missing.

Let player K start first. We note than the player K does not want to a move like this, or else he will lose.

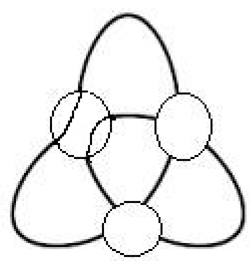


This is because it will give Player L the opportunity to perform this type of move,

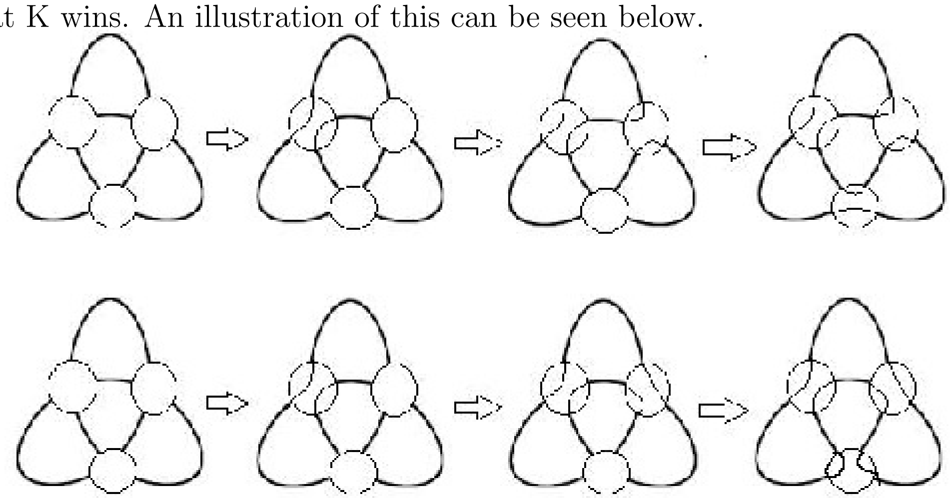


which will cause the configuration to be disconnected, which means Player L wins.

Thus, we see that player K must perform this move, as to not give the player L a winning advantage.

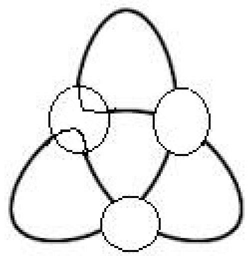


We see that from here, the player K has a winning strategy. This is because no matter what move L does, player K can perform a move such that K wins. An illustration of this can be seen below.

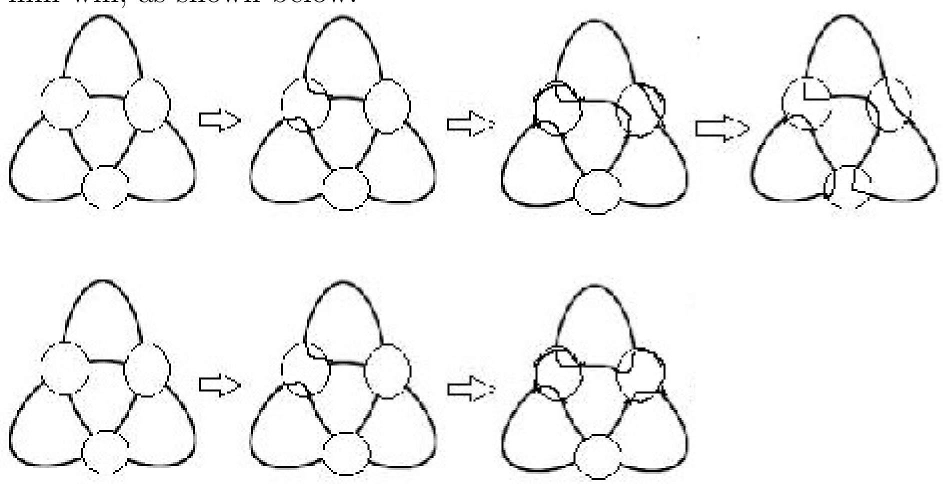


Thus, we see if player K goes first, then he has a winning strategy.

Now, let player L go first, and make this move.



Now, no matter what player K does, player L can perform a move that lets him win, as shown below.

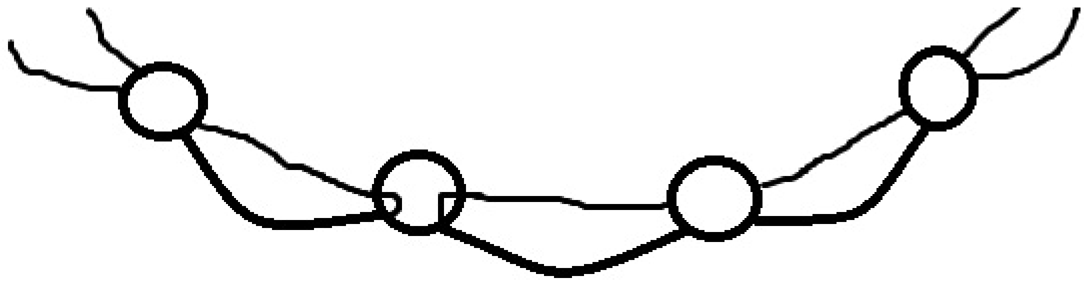


Thus, we see that if player L goes first, then he has a winning strategy. From this, we see that the first player has a winning strategy.

3.2 Braid knots of more than 3 crossings

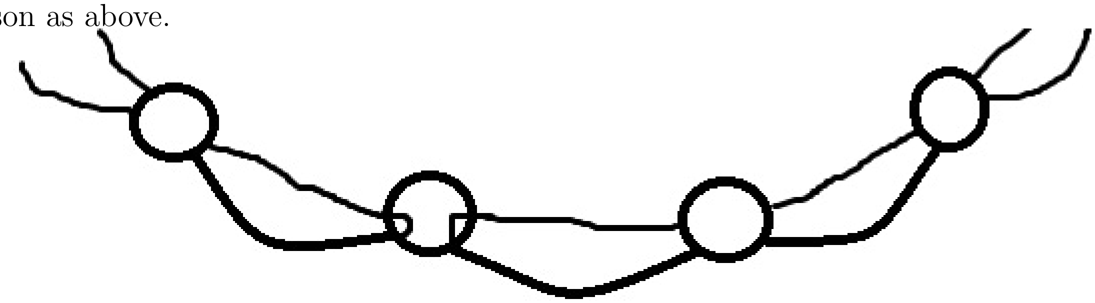
Claim: Player L has a winning strategy for braid knots with more than 3 crossings.

Proof. Let player L go first, and make this type of move.



Now, no matter what player K does, player L can make the move again and win. This is because all L needs to have two of these moves, in that by placing 2 of these moves, he effectively separates the part of the knot in between these two moves from the rest of the knot. Thus, L has a winning strategy if he goes first.

Now, let player K go first. Since there are 4 or more crossings, both players would have at least 2 turns When it's player L's turns, he only needs to places two of these knots. This would ensure his victory, for the same reason as above.

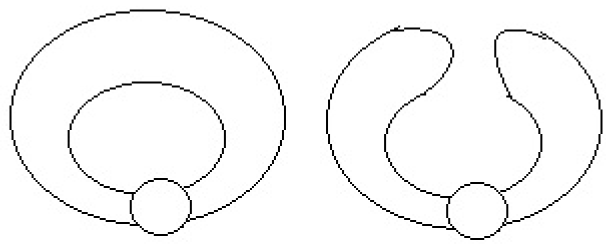


□

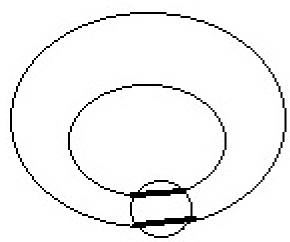
3.3 Some other strategies

Proposition 1. *If the Link plays last in the game, then the link wins*, which means
(1). If the diagram has an even number of crossings and the Player L plays second, then L wins,
(2). If the diagram has an odd number of crossings and the Player K plays first, then L wins.

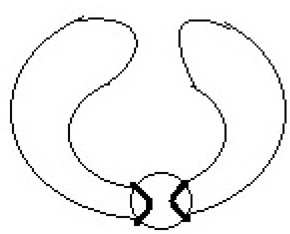
Proof. If L does not win on his moves before the last move, then there are only two projection cases before the last move. Since there's only smoothing move, other cases can only be some disconnected circles which means L has already won before the last move.



In the first case, if L does horizontal smoothing, then it will get two circles apart— the unlink, which means the link player wins.



In the second case, if L does vertically smoothing, then it will get the unlink, too. Thus the Link player wins.

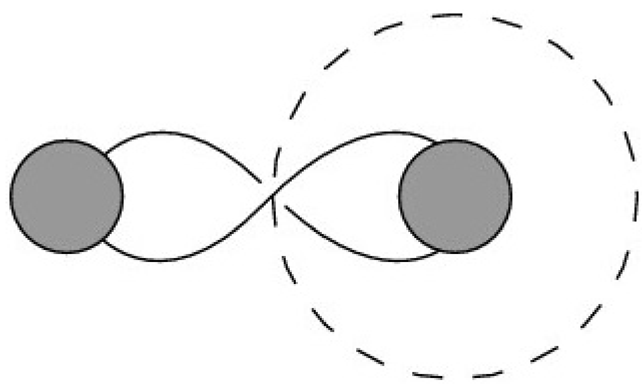


It is also obvious that the two cases above are actually the same.
Any diagram with one remaining pre-crossing can be resolved so that the number of components either increases from one to two or remains at two or greater. □

We should also notice that player K can only win on the last move, but player L can win before the last move. Because we cannot make sure the diagram stays in one piece until the last move is made, but we can actually split the diagram apart before the last move.

Proposition 2. Suppose L plays first. Then *L has a winning strategy if the diagram contains a reducible crossing, i.e. a crossing that when appropriately smoothed disconnects the shadow.*

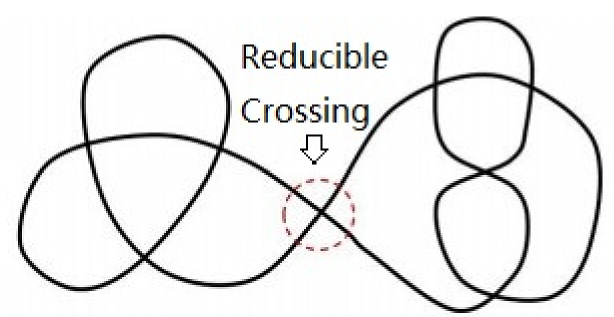
Reducible Crossing: A crossing in a knot diagram for which there exists a circle in the projection plane meeting the diagram transversely at that crossing, but not meeting the diagram at any other point.

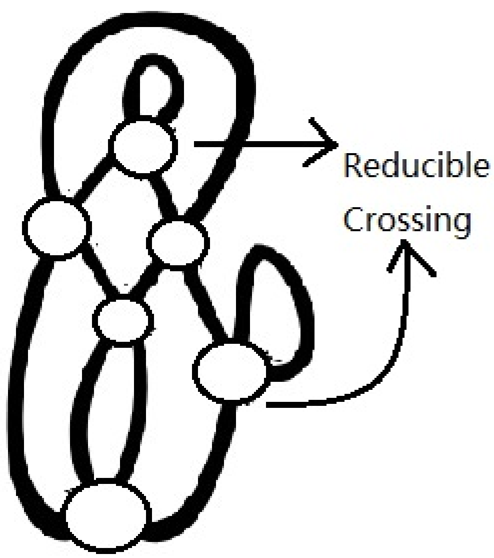


reducible crossing

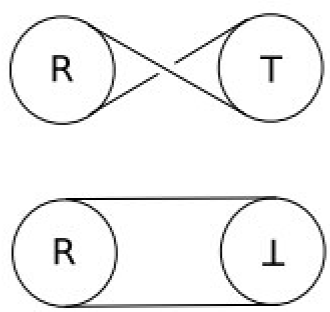
Proof. If a reducible crossing is present, L can win on her first move. If L vertically smooths the middle crossing of the triple, then she can win on her second move. □

Some example of reducible crossings.



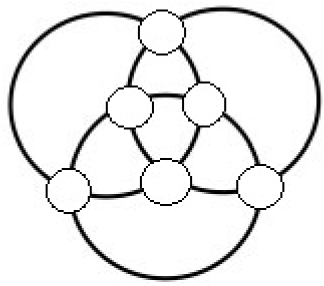


Thus for every reducible crossing, we can do the following step:



By doing this continuously, in the end we can get a reduced knot diagram.
Reduced knot diagram: A knot diagram in which none of the crossings are reducible.

Does Link always have the upper hand??
No. This is an example of a diagram where Knot actually has a winning strategy if she plays second.



References:
Henrich, A., and Johnson, I. (n.d.). The Link Smoothing Game. Retrieved from http://arxiv.org/pdf/1109.4103v1.pdf
Link,Unlink, *Wikipedia.*