CCS Problem-Solving I — Professor: Padraic Bartlett

Homework 9: Number Theory (Basic)

Due Tuesday, Week 5, at the start of class. — *UCSB 2014*

Solve **three** of the following **six** problems. As always, prove your claims!

0. Solve any un-signed-up-for problems from HW#6!

1. (IMO, 1975.) Let $f(n)$ denote the function that sends n to the sum of its decimal digits. For example, $f(746) = 7 + 4 + 6 = 17$. Find

$$f\left(f\left(f\left(4444^{4444}\right)\right)\right).$$

   Hint: mod 9 may be useful.

2. Fix any positive integer k. Consider the following sequence$\{a_n\}_{n=1}^{\infty}$, defined recursively:

   - $a_1 = 1$.
   - a_n is the n-th positive integer that is greater than a_{n-1} that is also congruent to n mod k.

   For example, if $k = 2$, we would get the sequence

$$1, 4, 9, 16, \ldots$$

   because

   - 4 is the second positive integer greater than 1 that is congruent to 2 mod 2,
   - 9 is the third positive integer greater than 4 that is congruent to 3 mod 2,
   - 16 is the fourth positive integer greater than 9 that is congruent to 4 mod 2, ...

   For arbitrary k, find a closed formula for a_n.

3. Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001.$$

   (Hint: if you can just show that a is unique, then n's uniqueness will follow immediately. To get that a is unique: try rewriting the above expression slightly, and use that to get information about a!)

4. Find all functions $g : \mathbb{N} \to \mathbb{N}$ that satisfy the following property:

$$\forall n \in \mathbb{N}, g(n) + 2 \cdot (g(g(n))) = 3n + 5.$$

5. (USAMO, 1979) Find all of the non-negative integer solutions $(x_1, \ldots x_{14}) \in \mathbb{N}^{14}$ to the equation

$$n_1^4 + n_2^4 + n_3^4 + \ldots + n_{14}^4 = 1599.$$

6. Prove that there are infinitely many primes of the form $4n - 1$, where n is an integer.