CCS Problem-Solving I — Professor: Padraic Bartlett

Homework 4: Extremal Elements

Due Thursday, Week 2, at the start of class. — *UCSB 2014*

Solve **one** of the following **three** problems. As always, prove your claims!

The theme for these problems: sometimes, when solving a problem, it is useful to think about finding the **biggest** or **smallest** element to help create a proof by induction/recursion/contradiction/etc. These problems all have solutions that involve this idea.

0. Solve any un-signed-up-for problems from HW#3!

1. Show that it is impossible to divide up a cube into finitely many smaller cubes, if we add the restriction that all of these smaller cubes have different sizes.

2. We are at a Viking luncheon, where there are a number of vikings and a number of delicious sandwiches. Assume that no Viking likes every sandwich, and also that every sandwich is liked by at least one Viking. Prove that there are two Vikings v, w and sandwiches s, t such that

   - v likes s.
   - v does not like t.
   - w likes t.
   - w does not like s.

3. Somewhere out in the vast interstellar void of space, suppose that we have n spherical planets, all with the same radius. Also assume that none of our planets intersect, because that would be awkward; also assume that all of our planets are fixed (i.e. not moving or rotating,) and that there's nothing else in between any of our planets.

   On the surface of each planet, there may be some regions that cannot be seen from any of the other planets. (To give an illustrative example: from the Earth, you can only see about half of the moon at any point in time. If we added in Mars as well, we might be able to see more parts of the moon, but there still could be other parts we can't see.)

   Prove that the total surface area of all of these sets over all of our planets is equal to precisely the surface area of one planet.

   (Hint: pick any vector in space, and use it to define a notion of "north pole" for all of our planets. Use this notion to create an ordering on our planets: given two planets A, B, we define $A > B$ if and only if when we remove all of the other planets from space, we can see the north pole of B from the planet A.

   Show that this order has a unique maximum element M – i.e. a unique planet that is greater than all of the other planets. What can you say about its north pole?)