CCS Problem-Solving I — Professor: Padraic Bartlett

# Homework 17: Putnam Problems

*Due Tuesday, Week 10, at the start of class.* — *UCSB 2014*

Solve **three** of the following **six** problems. All problems here are taken from past Putnam exams. As always, prove your claims/have fun!

0. Solve any un-signed-up-for problems from HW#16!
1. Take any positive nonprime integer $n > 1$. Show that we can find three positive integers $a, b, c$ such that
$$n = ab + ac + bc + 1.$$
2. Take the unit sphere $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Pick five points $A, B, C, D, E$ on this sphere. Show that there is some way to cut this sphere in half, so that at least four of these points are on the same half of our sphere.
3. Suppose that $P, Q, R$ are three points in the plane with the following properties:
   - $P, Q, R$ have integer coördinates.
   - The three points $P, Q, R$ are not collinear: that is, there is no single line $L$ that contains all three of these points.
   - The distances $d(P, Q), d(P, R), d(Q, R)$ are all integers.

   What is the smallest possible value for the distance $d(P, Q)$?
4. Prove or disprove: there is no sequence $\{x_k\}_{k=1}^{\infty}$ of real numbers such that for every $n \in \mathbb{N}$, we have
$$\sum_{k=1}^{\infty} x_k^n = n.$$
5. Suppose that there is a platform off the coasts of the Catalina Islands outfitted with sensors that measure the tides[1]. Tides are fairly predictable, and over time scientists have noticed that if $t$ is the current time and $T_t$ is the height of the tides at time $t$, there is some fixed polynomial of degree at most 3, $p(x)$ such that for any $t$, we have $T_t = p(t)$.

   However, scientists have not yet determined what $p(x)$ is, and are facing budget cuts that will stop them from being able to continuously measure the tides at this platform.
   (a) Show that without knowing what $p(x)$ is, we can find two times $t_1 < t_2$ such that the average tidal height from 9am to 3pm is $\frac{p(t_1)+p(t_2)}{2}$.
   (b) Show that $t_1 \approx$10:15am, $t_2 \approx$ 1:45pm.
6. Find all polynomials $p(x)$ such that $f(0) = 0$ and for all $x$, $f(x^2 + 1) = (f(x))^2 + 1$.

[1] The tide $T_t$ at any time $t$, for this problem, is simply the distance from the sea floor to the platform.