CCS Problem-Solving I | Professor: Padraic Bartlett

Homework 1: Introduction / Russian Problems

Due at the start of the next class. | *UCSB 2014*

Solve as many as you can! Instructions are in the syllabus. Prove any claims you make.

1. Is $\sin(10°)$ a rational number?
2. Which number is larger: $\log_2(3)$ or $\log_3(5)$? (For this problem, simply using Mathematica is not enough; you need a proof that doesn't use any computer-aided algebra systems.)
3. Take an arbitrary quadrilateral. In how many ways can you represent it as the union of two triangles? (Hints: break your quadrilateral into two cases, depending on whether or not your polygon is convex.)
4. Can you find an equilateral triangle T in the plane such that all three vertices of T have integer coördinates?
5. Find all of the functions $F : \mathbb{R} \to \mathbb{R}$ that satisfy the following property:

$$\text{For all } x, y \text{ in } \mathbb{R}, \quad F(x) - F(y) \leq (x - y)^2$$

6. Call a pair of integers a, b **intermingled**[1] if they satisfy the following properties:

   (a) a and b are distinct.

   (b) The prime decompositions of a and b share the same primes. In other words: if p is a prime that divides a, then it divides b. As well, if p is a prime that divides b, then it divides a as well.

   (c) The prime decompositions of $a + 1$ and $b + 1$ also share the same primes.

   For example, $(2, 8)$ satisfies this property; 2 and 8 both have only 2's in their prime factorizations, while 3 and 9 both only have 3's. $(6, 48)$ is another pair.

   Are there infinitely many such pairs?

[1] I made up this term.