CCS Discrete II | Professor: Padraic Bartlett

Lecture 5: Trees!

Week 5 | *UCSB 2015*

We closed our last lecture with a discussion of bipartite graphs. In that discussion, we came up with a really nice characterization (a graph is bipartite if and only if it doesn't have a subgraph isomorphic to an odd-length cycle) for this entire family of graphs – a powerful result!

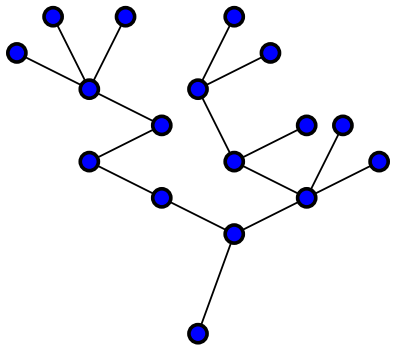
In graph theory in general, proving powerful results like the above can only happen when we're focusing on a specific family of graphs. This is not to say that we **can't** come up with properties that are universal to all graphs — in our "proof" of the four-color theorem, we showed that summing up the degrees of all of the vertices in any graph will always give you twice the number of edges. However, because there are **so many** different kinds of graphs, coming up with really interesting statements about all graphs will usually be impossible. Therefore, what we'll usually do in this course is study specific families of graphs – like, say, bipartite graphs! – and come up with specific properties for those families!

In today's lecture, we turn our attention to the family of **tree** graphs:

1 Basic Properties

Definition. If a graph G has no subgraphs isomorphic to cycles, we call G **acyclic**. A **forest** is another word for an acyclic graph; similarly, a **tree** T is a graph that's both connected and acyclic. In a tree, a **leaf** is a vertex whose degree is 1.

Example. The following graph is a tree:



Trees have a number of properties, which we quickly state here:

Proposition 1. *If G is a tree, G is bipartite.*

Proof. We proved in our last lecture that a graph is bipartite if and only if it has no subgraphs isomorphic to cycles of odd length. If G is a tree, none of its subgraphs are isomorphic to cycles of any length at all (and in specific none of the odd ones;) therefore, G is bipartite, as claimed. □

Proposition 2. *Every finite tree T with at least two vertices has at least two leaves.*

Proof. Take any tree T on n vertices, and look at all of the paths in T that don't repeat any vertices. Because there are finitely many vertices and edges in T, there are finitely many such paths; therefore, there must be a **maximum length** M that these paths can reach. Let P be such a path, and let v_0, v_M be its two endpoints.

We claim that these two endpoints are both leaves. Indeed, if one of them wasn't a leaf, then there would have to be at least two edges leaving that vertex, one of which was not used by our maximal path P. If that edge went back to some other vertex in P, it would make a cycle (which our graph T doesn't have, as it's a tree); if it went to some vertex that's not in P, adding that edge and vertex to our path would a path of length $M + 1$, which is strictly longer than our maximum-length path P (and is therefore impossible.) Thus, both of these endpoints are leaves. □

Proposition 3. *Deleting a leaf from a tree on n vertices produces a new tree on $n - 1$ vertices.*

Proof. Take any tree T on n vertices, let $l \in V(T)$ be a leaf of T, and let $u, v \in V(T)$ be a pair of distinct vertices, neither of which are L.

Because T is connected, there must be a path P from u to v: pick this path P so that it doesn't repeat any vertices or edges. (You showed that we can do this on the HW.) We know that this path cannot involve l (because if it went to l, it would have to travel along the one edge that goes to l twice); therefore, if we delete l and its edge from our graph, we still have a path from u to v. Therefore, the graph $T \setminus \{l\}$ is still connected. Because deleting an edge cannot create a cycle, this means that the graph $T \setminus \{l\}$ is a tree. □

Theorem. For a graph G on n vertices, the following statements are equivalent[1]:

1. G is a tree.
2. G is connected and has $n - 1$ edges.
3. G has $n - 1$ edges and no cycles.
4. G is a connected graph, and every edge of G is a cut-edge[2].

Proof. HW! □

2 How Many Trees Exist on n Vertices?

Given a property – say, being bipartite, or being a tree – a question we often like to ask in graph theory (and the field of combinatorics in general!) is *how many* objects there are that satisfy that property. For example, one question we might ask is the following:

[1] A series of true-false statements are called **equivalent** if one of them being true means that all of the others are true. For example, the two statements "n is odd" and "$n + 1$ is even" are equivalent: whenever one of them is true, the other must be true as well.

[2] An edge $e \in G$ is called a **cut-edge** if deleting e from G increases the number of connected components of G.

Question 4. *How many distinct trees are there on n vertices?*

(By "distinct," we mean "distinct as graphs" – i.e. we will regard two trees as being the same if and only if they are identical, not if they're just isomorphic to each other. To denote this, we will say that we're asking for "distinct labeled trees," to emphasize that we're caring about these as graphs, not just as graphs up to isomorphism.)

There are a number of beautiful proofs that answer this question (one of which we'll see later!) We study one such proof here:

Theorem 5. *(Cayley) There are n^{n-2} distinct labeled trees on the vertex set $\{1, 2, \ldots n\}$.*

Proof. A trick we often employ in mathematics is the art of counting a quantity in two ways (like in the degree-sum proof.) This trick is how we'll prove this claim: specifically, we're going to show that there is a bijection[3] between

- the collection of trees on $\{1, 2, \ldots n\}$, and
- the sequences of length $n-2$ of numbers from the set $\{1, 2, \ldots n\}$.

Because there are n^{n-2} such sequences (this is pretty easy to see: each entry in our sequence has n choices and we're picking $n-2$ entries,) if we can find such a bijection, it will prove that there are n^{n-2} trees.

How can we turn a tree into a sequence $\{a_1, \ldots a_{n-2}\}$ of numbers $1 \ldots n$? The trick, here, is the following ingenious algorithm discovered by the mathematician Heinz Prüfer in 1918:

1. As input, take a tree on n vertices, with its vertices labeled with the numbers $\{1, \ldots n\}$.
2. Look at the leaves of our tree; these are all labeled with distinct numbers. Let l be the leaf of our tree with the smallest possible label, and let v be l's only neighbor.
3. Delete l from our tree, mark it as "finished," and write down v's label as the first entry in our sequence.
4. If our tree doesn't consist of a single edge, repeat this process! I.e. go to step 2, where we look at all of the leaves and pick the one with the smallest label, then to step 3, where we delete this leaf and put the label of its only neighbor as the next entry in our sequence, then return here again.
5. If our tree does consist of a single edge, then only two vertices remain (both of which are not marked "finished,") and we've created a sequence of length $n-2$. Stop.

This process turns trees into sequences of numbers in $\{1, \ldots n\}$ of length $n-2$. To show that it's a bijection, we simply need to create an **inverse** process to the Prüfer algorithm: i.e. a process that will take any sequence $(a_1, a_2, \ldots a_{n-2})$ of elements from $\{1, \ldots n\}$ and

[3] A **bijection** between two collections A, B of objects is a map that sends each element of A to an element of B, in a way such that each element of B is matched to exactly one element of A. Intuitively, a bijection is just a way of "matching up" elements of A and B. Notice that if there's a bijection between A and B, then A and B have the same number of elements (because each element of A has a corresponding unique element in B, and vice-versa.

create a tree, such that putting that tree into the Prüfer algorithm will return the same sequence.

To do this, let $A = (a_1, a_2, \ldots a_{n-2})$ be any sequence of numbers from $\{1, \ldots n\}$. As well, create a list $\mathcal{E} = (e_1, \ldots e_n)$ of integers, where each e_i is equal to the number of times the number i occurs in A, plus one. (The idea of this list $\mathcal{E}$ is that it is counting the number of edges that each node i in our tree will be incident with.) We then proceed by the following algorithm:

1. Let i be the first non-struck-out entry in A. Strike out the first entry from A.
2. Let j be the smallest index in $\{1, \ldots n\}$ such that $e_j = 1$. Mark j as "finished," draw an edge connecting i to j, and decrement e_i, e_j both by 1. (Question: why is this operation well-defined? Prove it to yourself!)
3. If there are any non-zero indices in $\mathcal{E}$, go to 1 and repeat this process.
4. Otherwise, there are no elements left in A, and thus only two nonzero elements left in $\mathcal{E}$, both of which are 1/neither of which are finished. Connect these two elements with an edge.

To see that this process always creates a tree T, simply notice that

- the process above starts off with T as a graph where each connected component contains an unfinished vertex, and
- each stage consists of joining two unfinished vertices in distinct components and marking one vertex as finished: this decreases the number of connected components by 1 and leaves one unfinished vertex in each connected component.
- Because the process above runs for $n-1$ steps, creates $n-1$ edges in T, and we start with n connected componenets, it produces a graph with $n-1$ edges on n vertices that's connected. By our earlier proposition, this is a tree.

Finally, to prove our claim, it suffices to show that these two algorithms are inverses of each other: i.e. that taking any sequence, applying the above inverse map, and then applying the Prüfer map will result in the same sequence.

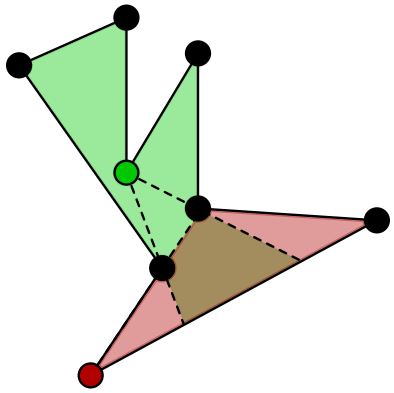
Proving this is a HW exercise! (But it's not hard: consider those "finished" labels we attached to vertices in the Prüfer algorithm and its inverse step.) □

With the rest of this talk, we turn to an application of our understanding of how trees work:

3 The Art Gallery Problem

Consider the following question:

Question 6. *Suppose that you have an art gallery that is shaped like some sort of n-polygon, and you want to place cameras with $360°$-viewing angles along the vertices of your polygon in such a way that the entire gallery is under surveillance. How many cameras do you need?*



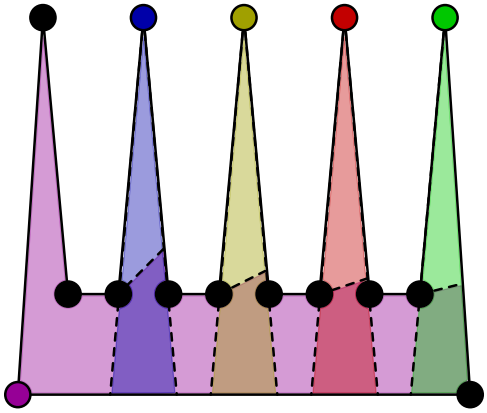
A gallery guarded by 2 guards, Red and Green.

One trivial upper bound you can come up with is n guards: just put one guard on each vertex of our polygon with n sides!

Can we do better? As it turns out, we can!

Claim. (Chvátal) You need at most $\lfloor n/3 \rfloor$-many cameras to guard a n-polygon.

It bears noting that this bound of $\lfloor n/3 \rfloor$ is sharp. Consider the following art gallery:



A crown-shaped art gallery.

In this sort-of "crown-shaped" art gallery, each prong of the crown (i.e. triangle) needs to have a guard on one of its three vertices to guard the entire triangle, as no other vertices can "see" the entirety of that prong. Therefore, you need one guard for each prong; i.e. $n/3$ guards, for a crown with $n/3$ prongs (i.e. n vertices.)

To prove Chvátal's theorem, we need a few lemmas first:

Lemma 7. *If G is a n-polygon with $n \geq 4$, then there is some line segment formed by two of the vertices in G that lies entirely in G.*

Proof. Let v be the leftmost vertex of G. (If there is a tie, take v to be the top leftmost vertex of G.) Let u and w be v's neighbors, and examine the line segment $\overline{uw}$. If this lies entirely in G, great! Otherwise, it must cross some edge of G; consequently, there must be a vertex of G that lies inside of the triangle spanned by the three points u, v, w. Let x be

the vertex furthest from the line segment $\overline{uw}$ that lies in this triangle. Then, look at the line segment $\overline{vx}$; because x is the furthest point in Δuvw from $\overline{uw}$, there can't be any edges of G that are crossed by this line segment (as one of their endpoints would necessarily be closer to v.) So $\overline{vx}$ lies entirely in G. □

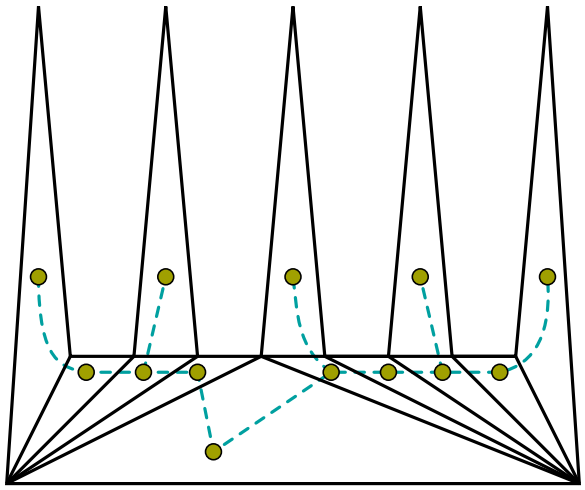
Corollary 8. *Any n-polygon can be divided into $n-2$-triangles.*

Proof. Using the process above, repeatedly divide our n-polygon into a pair of smaller polygons, one with k vertices and the other with $n+2-k$ vertices, until all of these polygons are triangles. By induction, it is not hard to see that the number of these triangles is $n-2$. □

So: we can turn any polygon into a number of triangles stuck to each other! We use this to turn any art gallery on n vertices into a **graph** on $n-2$ vertices, as follows:

- Start by taking our polygon G and turning it into a collection $\{T_i\}_{i=1}^{n-2}$ of triangles.
- For each triangle T_i, associate a vertex t_i.
- Connect two vertices t_i, t_j with an edge if their corresponding triangles T_i, T_j share a face.

Call this graph T' the **dual graph** of T.



Turning the crown into a tree.

This is a graph! Furthermore, it's a fairly special kind of graph: it's a tree! We prove this here:

Lemma 9. *Let G be a polygon, T be a triangulation of G performed as above, and let T' be the dual graph to this triangulation (i.e. put a vertex in the center of every face of T , and connect two faces iff they share an edge.) This graph is a tree.*

Proof. Let T be our triangulated polygon. In our construction above, each of the edges of T' corresponds to a diagonal of the polygon G, that divides our polygon into two distinct smaller polygons. Because cutting our polygon G along one of those diagonals will always divide the polygon into two disconnected pieces, doing so will always result in two triangles that are no longer connected by a chain of triangles with adjacent faces!

In other words: in the dual graph T' that we made above, removing any edge disconnects our graph! So our graph is a tree, by our theorem from earlier. □

This tree is remarkably useful; in particular, we can use its structure to create a system for assigning guards! We do this here:

Lemma 10. *Take a polygon G that has been triangulated as described earlier. Then we can color each of the vertices of G either red, blue or green, so that each triangle contains one vertex of each color.*

Proof. For our triangulated polygon G, take the dual graph/tree T' that we constructed above, and pick some vertex t_0 in it. For all relevant integers k, let T'_k be the collection of vertices that are distance k away from v, for every k. (The **distance** of two vertices from each other is the length of the shortest path between them.)

Color the vertices of T as follows:

- Take the triangle in G associated to t_0 and color its three vertices red, green and blue.
- Suppose we've colored all of the vertices attached to triangles with corresponding vertices in T'_i, for some i. Now, look at the triangles corresponding to vertices in T'_{i+1}. Each triangle associated to a vertex t_{i+1} in this set shares exactly one edge with some triangle associated to a vertex in T'_i; this is because if our vertex is distance $i+1$ from t_0, then (by taking the path of distance $i+1$ and walking one step closer to t_0) there is an adjacent vertex (and thus face-sharing triangle) at distance i, i.e. in T'_i.

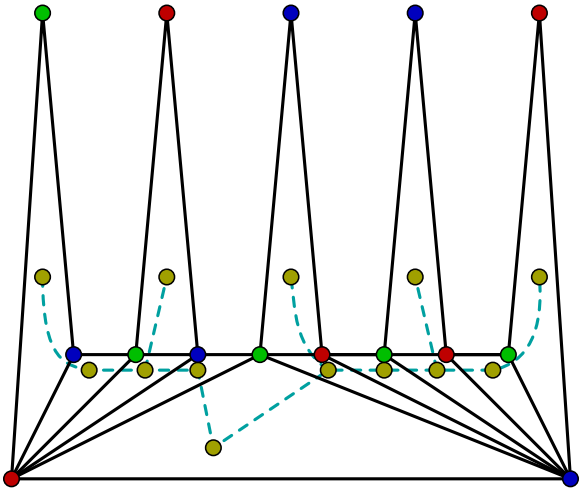
  Furthermore, because T' is a tree, there is exactly one edge from any t_{i+1} to vertices in the set $\bigcup_{j=0}^{i+1} T'_j$. This is because the existence of any other edge would create a cycle, because it would give us two distinct paths to t_0!

  Therefore, the triangle associated to t_{i+1} shares a face with **only one other** triangle in all of the sets that we've already colored! Therefore, only two of its vertices have been assigned colors. Thus, there is always some spare third color to use to color its third vertex! Use this to color its third vertex, and repeat for all vertices in T_{i+1}.
- Repeat until every vertex in T is colored. Note that each triangle has one vertex of each color.

□

Corollary 11. *You need at most $\lfloor n/3 \rfloor$-many cameras to guard a n-polygon.*

Proof. By the above, create a triangulation and 3-coloring of our polygon G with the colors $\{R, G, B\}$. Now, station guards at whichever color is used the least number of times in this triangulation! Each guard can see everything in their assigned triangle by construction. Therefore, the entire art gallery is guarded. □



To guard this crown, simply pick one of (red, green, blue,) and station guards at vertices of that color.