CCS Discrete II	Professor: Padraic Bartlett

Homework 8: Trees

Due Friday, Week 5	*UCSB 2015*

Do **one** of the four problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. Prove the tree theorem we mentioned in class:

   Theorem. For a graph G on n vertices, the following statements are equivalent[1]:

   - G is a tree.
   - G is connected and has $n - 1$ edges.
   - G has $n - 1$ edges and no cycles.
   - G is a connected graph, and every edge of G is a cut-edge[2].

2. For any graph G, let $\delta(G)$ denote the minimum degree over all of the vertices in G. Suppose that T is a tree on n vertices, and that G is a graph with $\delta(G) \geq n$. Then there is a subgraph of G isomorphic to T.

3. Suppose that T is a tree. As noted in class, T is bipartite. Let V_1, V_2 denote a bipartition of T's edges; i.e. $V_1 \cup V_2 = V(T)$, and every edge of T has exactly one edge in V_1 and another in V_2.

   Suppose that $|V_1| \geq |V_2|$. Prove that T has at least one leaf in the larger of the two sets V_1, V_2.

4. Prove the claim we made in class on Monday: that the Prüfer algorithm's inverse is in fact an inverse! In other words, prove that taking any tree T, running the Prüfer algorithm on it to get a sequence, and running the claimed inverse map to get a graph G will always return the same tree T.

[1] A series of true-false statements are called **equivalent** if one of them being true means that all of the others are true. For example, the two statements "n is odd" and "$n + 1$ is even" are equivalent: whenever one of them is true, the other must be true as well.

[2] An edge $e \in G$ is called a **cut-edge** if deleting e from G increases the number of connected components of G.