CCS Discrete II — Professor: Padraic Bartlett

Homework 7: Graphs (Fundamentals)

Due Friday, Week 5 — *UCSB 2015*

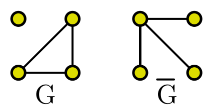
Do **three** of the five problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. (a) Suppose that G is a graph on n vertices with m edges, where $n \geq 4$ and $m > \frac{n^2}{4}$. Prove that G contains an odd-length cycle as a subgraph.

   (b) For any even n, create a graph on n vertices with $\frac{n^2}{4}$ edges that does not contain an odd-length cycle as a subgraph.

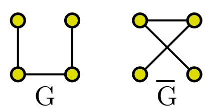
2. Given a graph G, its **complement** is the graph $\overline{G}$ formed as follows:

   - The vertices of $\overline{G}$ are the same as the vertices of G.
   - We connect two vertices in $\overline{G}$ with an edge if and only if they are not connected by an edge in G.



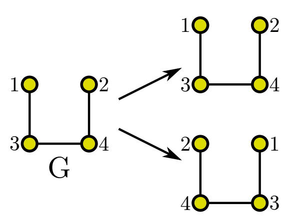
   (a) Show that for any graph G, at least one of $G, \overline{G}$ are connected.

   (b) A graph is called **self-complementary** if G is isomorphic to $\overline{G}$.



   Prove that if n is a multiple of 4, there is a self-complementary graph on n vertices.

3. Take a graph G. An **automorphism** of G is any isomorphism from a graph to itself. For example, take the graph G from above. It has two automorphisms: one where we send $v(1) = 1, v(2) = 2, v(3) = 3, v(4) = 4$, and another where we send $v(1) = 2, v(2) = 1, v(3) = 4, v(4) = 3$.



   (a) Prove that the two automorphisms listed above are the only automorphisms of the graph G as drawn.

(b) Take any graph G, and let φ, ψ denote a pair of automorphisms of G (i.e. a pair of isomorphisms from G to G.) Prove that $\varphi \circ \psi$, the map created by composing these two functions, is also an automorphism.

(c) Let $Aut(G)$ denote the collection of all of the automorphisms from a graph to itself. Prove that $Aut(G)$ is a group.

4. Take a connected graph G. We defined a notion of **distance** on the vertices of a graph last week as follows: for any two vertices $x, y \in V(G)$, we say that $d(x, y)$ is equal to the length of the shortest path in G that connects x and y.

   Prove that this notion of distance is a metric.

5. Given a connected planar graph G, we can form the **dual** to this graph, G^*, as follows:

   - Vertices of G^*: the faces of G.
   - Edges of G^*: connect two faces F_1, F_2 if they share an edge in common.

   (a) Prove that G^* is a connected planar graph.

   (b) Show that the dual of G^*, i.e. $(G^*)^*$, is just the graph G again.

   (c) Draw the five platonic solids as planar graphs.

   (d) Find the dual of each solid.