CCS Discrete II — Professor: Padraic Bartlett

Homework 3: Vector Spaces and Metrics

Due Friday, Week 2 — *UCSB 2015*

Do **one** of the **three** problems below! Prove all of your claims.

1. (a) Let $\mathcal{C}(\mathbb{R})$ denote the collection of all continuous functions on the real numbers. Is this a vector space over $\mathbb{R}$? Prove your answer.

   (b) Let $\not\mathcal{C}(\mathbb{R}) \cup \{0\}$ denote the collection of all discontinuous functions on the real numbers, along with the identically-0 function $f(x) = 0$. Is this a vector space over $\mathbb{R}$? Prove your answer.

2. Given a field $\langle F, +, \cdot \rangle$, we say that a subset $G \subseteq F$ is a **subfield** of F if $\langle G, +, \cdot \rangle$ is a field in its own right (if we interpret $+, \cdot$ here as the same operations that we used on F.) For example, $\mathbb{Q}$ is a subfield of $\mathbb{R}$, and $\mathbb{R}$ is a subfield of $\mathbb{C}$. Conversely, $\mathbb{Z}/2\mathbb{Z}$ is not a subfield of $\mathbb{Q}$; even though 0 and 1 are both elements of $\mathbb{Q}$, the addition operation on $\mathbb{Z}/2\mathbb{Z}$ does not agree with the addition operation on $\mathbb{Q}$.

   (a) Find a field $\langle F, +, \cdot \rangle$ whose only subfield is itself.

   (b) Show that if $\langle F, +, \cdot \rangle$ is a field and G is a subfield of F, then F is a **vector space** over the field G.

   (c) Show that not all vector spaces have this structure: i.e. find a vector space V over a field F such that F is not a subfield of V.

3. Consider the following metric, called the "post office" metric, on $\mathbb{R}^n$:

   - For any $\vec{x} \in \mathbb{R}^n$, we have $d(\vec{x}, \vec{x}) = 0$.
   - For any $\vec{x} \neq \vec{y} \in \mathbb{R}^n$, we have $d(\vec{x}, \vec{y}) = ||\vec{x}|| + ||\vec{y}||$. (If you haven't seen this notation before, $||\vec{x}||$ denotes the quantity $\sqrt{x_1^2 + \ldots x_n^2}$.)

   The idea with this metric is that if you want to mail a package from some location $\vec{x}$ to another location $\vec{y}$, you often have to route it through some central packing location (in this case, $\vec{0}$.)

   Prove that the post office metric is a metric.