CCS Discrete II	Professor: Padraic Bartlett

Homework 15: Random Graphs

Due Friday, Week 10	*UCSB 2015*

Do **three** of the six problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. More Sim variations! Consider the following game, which we can think of as **impartial** or **colorblind** Sim:

   - There are two players, Player 1 and Player 2. Their gameboard consists of n points drawn on a plane. Players alternate turns, and Player 1 starts first.
   - On a given player's turn, they must connect two points that do not have a line drawn between them yet. All lines are drawn in black.
   - You lose if your chosen move on any turn creates a triangle.

   This is like Sim if you were colorblind (and thus to be safe had to avoid all triangles.)

   Who wins[1] this game on $n = 5$?

2. Yet more Sim variations! Consider the following game, which we can think of as **misère impartial** or **colorblind** Sim:

   - There are two players, Player 1 and Player 2. Their gameboard consists of n points drawn on a plane. Players alternate turns, and Player 1 starts first.
   - On a given player's turn, they must connect two points that do not have a line drawn between them yet. All lines are drawn in black.
   - You **win** if your chosen move on any turn creates a triangle.

   This is like misère Sim if you were colorblind.

   Determine who wins this game for **all** values of n!

3. Let $\mathcal{F}(\mathbb{N})$ denote the collection of all finite subsets of $\mathbb{N}$. Prove that $\mathcal{F}(\mathbb{N}) \times \mathcal{F}(\mathbb{N})$ is a countable set.

4. Let $\langle \Omega, Pr \rangle$ be some probability space, and $A_1, A_2 \ldots A_k$ a collection of mutually independent events in this space.

   (a) Prove that $Pr(A_1 \cap A_2 \cap \ldots \cap A_k) = \prod_{i=1}^{k} Pr(A_i)$.

   (b) For any event A_i, let $A_i^c = \{\omega \in \Omega \mid \omega \notin A_i\}$; i.e. A_i^c is the **complement** to A_i. Prove that the events $A_1^c, \ldots A_k^c$ are all mutually independent.

[1] Warning: in general, this problem is open for general values of n as far as I know. See "One-color Triangle Avoidance Games" and "On Hajnal's Triangle-free Game" for more information. So a general solution for all n is probably very hard, though if you can find it that would be really exciting!

5. Take the Rado graph R from Wednesday, week 9's lecture. Delete finitely many vertices and edges from this graph, to get some new graph R'. Is R' isomorphic to R? If it is, prove your claim; if it is not, give an example that proves that this can fail.

6. Consider the graph $\mathcal{B}$ on the vertex set $\mathbb{N}$, formed by drawing an edge $\{x, y\}$ between two numbers x, y whenever either the x-th bit of y's binary representation is 1, or the y-th bit of x's binary representation is 1. So, for example:

   - The two vertices $46, 3$ would be connected, as $46 = 101110_{\text{binary}}$, and the third bit of this sequence is a 1.
   - Similarly, the two vertices $19, 4$ would not be connected; $19 = 10011_{\text{binary}}$, the fourth bit of which is 0, and similarly $4 = 100_{\text{binary}}$, and the 19th bit[2] of that sequence is also 0.

   Show that this graph is isomorphic to the Rado graph.

[2]By the "19th bit of 100_{binary}, we just mean the 19th bit of this number in binary, where we can naturally think of $100_{\text{binary}} = 000\ldots0100_{\text{binary}}$. In other words, adding leading zeroes doesn't change numbers, and lets us refer to the "n-th bit" of any number and always have that quantity be defined.