CCS Discrete II — Professor: Padraic Bartlett

Homework 14: The Return of Set 12

Due Friday, Week 9 — *UCSB 2015*

Three things with this set:

- Because it came out late, I'm considering it as both Friday and Monday's problem set! So there is no set that will come out Monday / you just have 3 problems to do this week.
- Many of the problems here may seem familiar; this is because several of them are variations on problems from set 12, with additional hints to make them more approachable!
- You cannot use results from set 12 on this set. If you successfully solved a problem on set 12 that has reoccurred on this set, then you should probably try a different problem. (If you tried a problem but it didn't work, feel free to retry that problem!)

Do **three** of the six problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. If you did problems 2 or 3 on set 12, you are probably tired of Sim. Accordingly, if someone were to challenge you to a game of Sim, you might want to instead **lose** as quickly as possible, just to get the game over with!

   As mathematicians, we can turn this into another game: **Miseré Sim**.

   - There are two players, Red and Blue. Their gameboard consists of n points drawn on a plane. Players alternate turns, and Red starts first.
   - On a given player's turn, they must connect two points that do not have a line drawn between them yet, with a line of their given color.
   - The game ends when a monochromatic triangle is drawn on the board, in which case that player **wins**, because they have ended the game!

   Show that the first player can force a "win" in this game whenever $n \geq 5$.

2. Take K_{10}, and color each of its edges red or blue. Prove that there are either two disjoint[1] red triangles or two disjoint blue triangles.

   Hints: there are many ways to approach this problem. One way that I took was the following:

   - Mimicking the first problem on set 12, show that there are at least 20 monochrome triangles in our two-colored graph.
   - Show that if all of the monochrome triangles are the same color (i.e. there are none of the other color,) then our claim holds.

[1] As before, two graphs G, H are disjoint if and only if they share no vertices or edges in common.

- Otherwise, by symmetry (because the names of the colors don't matter) there must be at least one red and one blue triangle. Show that you can specifically find a red and a blue triangle with a vertex in common. Call the union of those two triangles G.
- Look at the five vertices that are not members of G; call the graph on those five vertices H. If there is either a red or blue triangle among those five vertices, then our proof is done. If there is not, what can you say about H?
- Finally, use the structure you have established for G, H to prove by contradiction that there must be two disjoing monochrome triangles of the same color.

3. Show that for any red-blue two-coloring of K_{5n}'s edges, there are at least n disjoint red triangles or n disjoint blue triangles,

   Hint: use induction, with problem 2 as your base case.

4. For any n, find a red-blue two-coloring of K_{5n-1} such that it is impossible to find n disjoint red triangles or n disjoint blue triangles.

5. Define a **k-hypergraph** $G_k = (V, E)$ as follows:

   - Vertices: some set V.
   - Edges, or "hyperedges:" a collection of subsets of V of size k.

   A graph, in this sense, is just a 2-hypergraph. Hypergraphs are just the idea behind a graph, where we let edges be subsets of size k instead of 2.

   The complete k-hypergraph on n vertices, $K_n^{(k)}$, is just a graph on the vertices $\{1, 2, \ldots n\}$ along with all of the subsets of $\{1, 2, \ldots n\}$ of size k. A red-blue two-coloring of such a hypergraph is any way to assign each hyperedge one of the two colors red or blue. Such a two-coloring is called monochrome if all of the edges are the same color.

   Define the **hypergraph Ramsey number** $R(a, b; k)$ as the smallest value of n such that any red-blue coloring of $K_n^{(k)}$ has either a red $K_a^{(k)}$ or a blue $K_b^{(k)}$. Prove that $R(a, b; k)$ exists[2] for any a, b, k.

   Hint: prove that

$$R(a, b; k) \leq 1 + R\Big(R(a, b-1; k), R(a-1, b; k); k-1\Big).$$

6. Given a graph G, a **dominating set** is any subset S of G's vertices such that every vertex not in S has a neighbor in S.

   Suppose that G is a n-vertex graph in which every vertex has degree at least k, for some value k. Prove that G has a dominating set of size at least $n\frac{1 + ln(k+1)}{k+1}$.

[2] You can use this result to prove the theorem from problem 6 last week, as we discussed then!

Hint: Pick out a random subset S of G's vertices by flipping a coin for each vertex, and putting it in S if and only if the coin comes up heads. Suppose that your coin comes up heads with probability p, and tails with probability $1 - p$.

Let T be the collection of all vertices that are not in S that have no neighbors in S. $S \cup T$ is a dominating set. Can you find an upper bound on the expected value of $|S \cup T|$?