CCS Discrete II	Professor: Padraic Bartlett

Homework 13: The Probabilistic Method

Due Friday, Week 8 | *UCSB 2015*

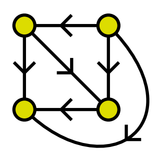
Do **one** of the three problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. Prove that for any two positive integers k, l, we have
$$R(k,l) \leq \binom{k+l-2}{k-1}.$$

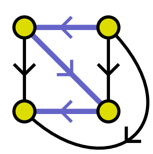
2. Take any subset B of n positive integers $\{b_1, \ldots b_n\}$. Using the probabilistic method, show that B contains a sum-free[1] subset of size $\geq n/3$.

   Hint: pick some prime p that's larger than twice the maximum absolute value of elements in B, and look at B modulo p (in other words, look at B as a subset of $\mathbb{Z}/p\mathbb{Z}$. Because of our clever choice of p, all of the elements in B are distinct mod p (why?) Now, look at the sets $xB := \{x \cdot b : b \in B\}$ in $\mathbb{Z}/p\mathbb{Z}$. Using the probabilistic method, show that there is some value of x such that more than a third of the elements of xB lie between $p/3$ and $2p/3$. Use this to attack your problem.

3. A n**-vertex tournament** is any way to take K_n and assign an orientation to each of its edges. For example, here is a tournament on 4 vertices:

   A **Hamiltonian path** on an oriented graph is any path that only travels on edges along their orientations, that visits each vertex exactly once. We draw an example of a Hamiltonian path on the graph above here:

   Prove that for any n, there is a n-vertex tournament with at least $\frac{n!}{2^{n-1}}$ different Hamiltonian paths.

[1] A subset S of $\mathbb{R}$ is called **sum-free** if for any $a, b \in S$, $a+b$ is not in S. For example, $\{1, 3, 5\}$ is sum-free.