CCS Discrete II	Professor: Padraic Bartlett

Homework 11: More Colorings

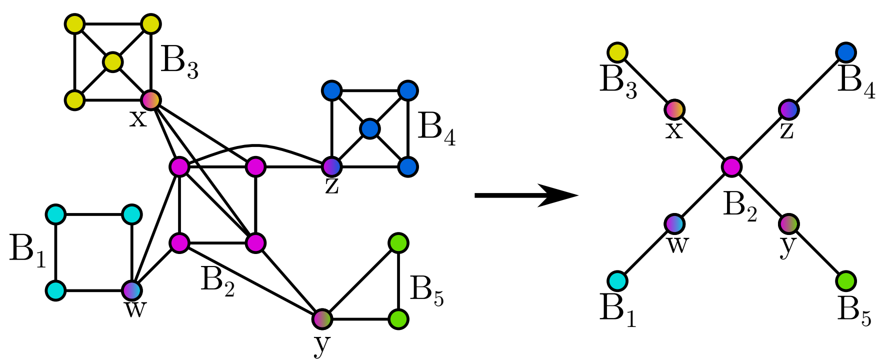
Due Friday, Week 7	*UCSB 2015*

Do **three** of the five problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. When we studied the Mycielski graph construction in class, we left one part of the proof for the homework: namely, the proof that if G is a graph with $\chi(G) = k$ and G' is the result of applying the Mycielski construction to G, then $\chi(G') > k$. Prove this claim!

2. In Wednesday's class, we introduced the idea of a **block-graph**, which we defined for a given graph G as follows:

   - A **block** subgraph B to be any subgraph that is 2-vertex connected and maximal: that is, there is no larger subgraph that is two-connected and contains B.
   - We define the **block graph** T_G of G as follows:
     - Vertices: create one vertex for each block B_i of our graph G, and another vertex of our graph for each cut-vertex x_j in our graph.
     - Edges: connect a block B_i to a cut-vertex x_j if and only if $x_j \in B_i$.

   A graph G along with its "block-tree" T_G.

   Suppose that G is a connected graph and T_G is its block-graph. Prove the following claims we made about block graphs:

   (a) Any two blocks in G share at most one cut-vertex in common.

   (b) If e is an edge linking two distinct blocks in G, then at least one endpoint of e is a cut-vertex.

   (c) The block-graph T_G is a tree.

3. (This problem was on HW 9, but it had a mistake that made it impossible — I wrote "graph" but wanted "bipartite graph." It's here now!) A **vertex cover** of a graph G is a subset of the vertices of G that contains at least one endpoint of every edge. Prove that for any bipartite graph G, the number of vertices in the smallest vertex cover is the same as the number of edges in the largest matching.

4. A graph G is called **perfect** if $\chi(G) = \omega(G)$.

   (a) Find a graph G that is perfect. Find another graph G' that is not perfect.

   (b) Show that if a graph G is bipartite, then its complement $\overline{G}$ is perfect.

5. Suppose that a graph G on n vertices has chromatic number k, where $k \geq 3$. Prove that G contains at most $\frac{n^2(k-1)!}{k}$ edges.