

Homework 2: Linear Algebra and More Circuits

Due Friday, Week 2

UCSB 2015

Do **three** of the following **five** problems!

1. (a) Prove the following “basis-trading lemma:”

Lemma. Suppose that V is a vector space, and that $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$, $C = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ are a pair of finite bases for V . Take any $\vec{v}_i \in B$. Then there is some $\vec{w}_j \in C$ such that the set

$$D = (C \cup \{\vec{v}_i\}) \setminus \{\vec{w}_j\}$$

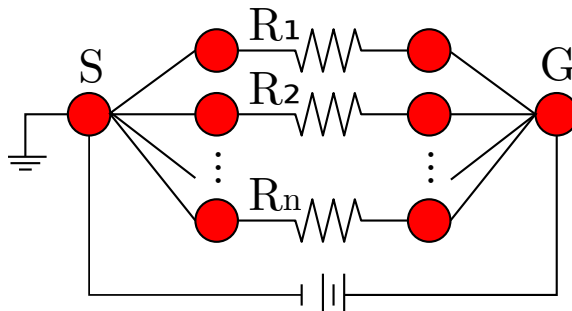
formed by “replacing” \vec{w}_j with \vec{v}_i is a basis.

- (b) Using (a), prove the following claim: if V is a vector space and B, C are two different bases for V , then $|B| = |C|$.
2. Using the Axiom of Choice, extend (1) above to infinite bases: that is, show that if V is a vector space and B, C are a pair of infinite bases for V , then $|B| = |C|$.
3. A basis for \mathbb{R}^n is called a **Hadamard basis** if it satisfies the following two properties:
 - ★. Every vector in the basis is made up out of entries from ± 1 .
 - ‡. The dot product of any two vectors in the basis is 0.

For example, $\{(1, 1), (1, -1)\}$ is such a basis for \mathbb{R}^2 .

- (a) Find such a basis for \mathbb{R}^4 .
 - (b) Show that no such basis exists for \mathbb{R}^7 .
 - (c) Show that such a basis exists for \mathbb{R}^n , whenever n is a power of 2.
4. Prove the following claims about resistors:

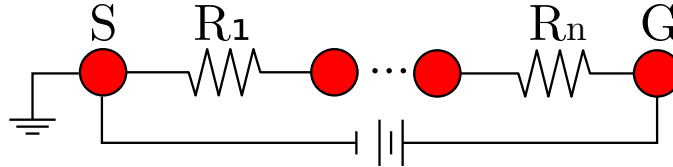
- (a) The effective resistance of the circuit below is the reciprocal of the sum of the reciprocals of the resistors in the circuit. In other words, the circuit



has effective resistance given by the formula

$$\frac{1}{R_{\text{eff}}} = \sum_{i=1}^n \frac{1}{R_i}.$$

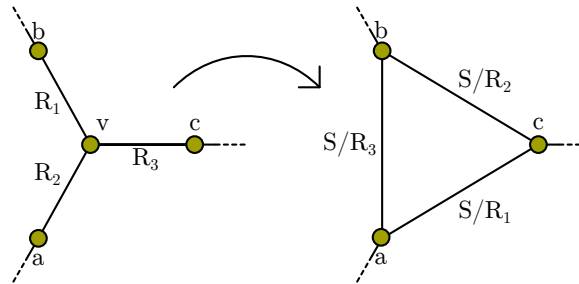
- (b) The effective resistance of the circuit below is the sum of the resistors in the circuit. In other words, the circuit



has effective resistance given by the formula

$$R_{\text{eff}} = \sum_{i=1}^n R_i.$$

5. Take an arbitrary circuit. Define the **star-triangle** transformation as follows:



The star-triangle transformation. In the above diagram, $S = R_1 R_2 + R_1 R_3 + R_2 R_3$.

This process takes a Y-configuration of resistors with resistances R_1, R_2, R_3 as labeled above, and replaces it with a Δ -configuration of resistors with resistances $S/R_1, S/R_2, S/R_3$ as labeled above, with $S = R_1 R_2 + R_1 R_3 + R_2 R_3$.

Prove that this replacement does not change the overall resistance of the circuit in which it is performed.

- (Bonus!) Show that there is a Hadamard basis for \mathbb{R}^n whenever n is a multiple of 4. (Warning: harder.)