CCS Discrete III	Professor: Padraic Bartlett

Homework 15: Spectral Graph Theory

Due Friday, Week 9	*UCSB 2015*

Choose **three** problems below to solve!

1. Given a $n \times n$ matrix A, the **trace** of A is the sum of the entries on the diagonal of A: that is,

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A(i,i).$$

   (a) Suppose that A is a $n \times n$ matrix that has n eigenvalues $\lambda_1, \ldots \lambda_n$ (counted with multiplicity.) Prove that $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

   (b) Let G be a finite graph with $n \times n$ adjacency matrix A_G and eigenvalues $\lambda_1, \ldots \lambda_n$ (where each eigenvalue is listed as many times as its multiplicity indicates.)
   Prove that $\lambda_1^2 + \ldots \lambda_n^2$ is a nonnegative integer. What very simple graph property does this sum correspond to?

2. Prove or disprove: there is no graph with $-1/2$ as an eigenvalue.

3. Let $K_{n,m}$ be the complete bipartite graph on (n, m) vertices (i.e. $K_{n,m}$ consists of two parts, one with n vertices and the other with m vertices. Every possible edge from one part to the other exists, and no edges exist within one given part.) Find the spectra of $K_{n,m}$.

4. Prove or disprove the following claim: there is some way to partition the edges of the graph K_{10} into three groups, each isomorphic to copies of the Petersen graph.

5. A **Latin square** is a $n \times n$ array filled with the symbols $\{1 \ldots n\}$, so that there are no repeated symbols in any row or column. Here is an example for when $n = 3$:

1	2	3
2	3	1
3	1	2

   Given any such square, form its corresponding **Latin square graph** as follows: create n^2 vertices, one for each cell in the square. Connect two vertices by an edge whenever one of the three following conditions hold:

   - The two cells corresponding to this vertex lie in the same row.
   - The two cells corresponding to this vertex lie in the same column.
   - The two cells corresponding to this vertex contain the same symbol.

   For any $n \geq 3$, prove that this graph is regular with degree $3(n-1)$. Show that this graph has precisely three distinct eigenvalues in its spectrum. What are they?