CCS Discrete Math I	Professor: Padraic Bartlett

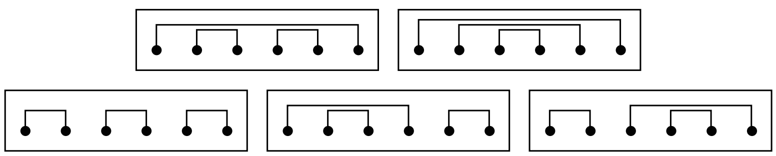
Homework 6: Catalan Numbers

Due Friday, Week 3 — *UCSB 2014*

This problem set is slightly different. Pick **two** of the **six** objects below, and show that they each satisfy the recurrence

$$C_0 = C_1 = 1, C_n = \sum_{k=0}^{n-1} C_k C_{(n-1)-k}.$$

1. Take $2n$ points in the plane, and pair them up by drawing nonintersecting arcs that lie above these points. Let C_n denote the total number of such pairs: we draw all of the configurations for $n = 3$ below.



   Show that the C_n's satisfy our claimed recurrence.

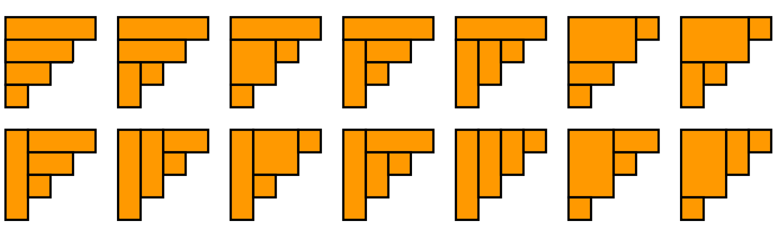
2. Take all of the sequences of integers $(a_1, a_2, \ldots a_n)$ such that
   - $1 \leq a_1 \leq a_2 \leq \ldots a_n$.
   - $a_i \leq i$, for every i.

   Let C_n denote the total number of such sequences of length n: we give all of the length-3 sequences here.

$$(1,1,1), (1,1,2), (1,1,3), (1,2,2), (1,2,3).$$

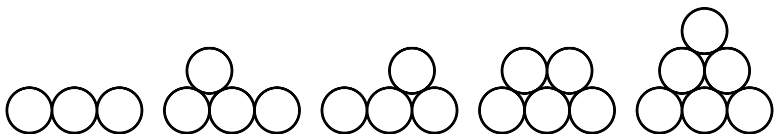
   Show that the C_n's satisfy our claimed recurrence.

3. A **stairstep** of height n is made by stacking a 1×1 block on top of a 1×2 block on top of a $\ldots 1 \times n$ block, to give us one of the diagrams below. A **tiling** of a stairstep by n rectangles is a way to cover one of these stairsteps with $k \times l$ rectangles, so that every block is covered and no block is covered twice. Let C_n denote the total number of coverings of a stairstep of height n with n rectangles: we draw the fourteen tilings of stairsteps of height 4 below.



Show that the C_n's satisfy our claimed recurrence.

4. A **valid coin-stacking** is any way to stack circles as drawn below, so that the bottom row consists of n consecutive coins. Let C_n denote the total number of valid coin-stackings such that the bottom row consists of n consecutive coins: we draw all of the configurations for $n = 3$ below.



Show that the C_n's satisfy our claimed recurrence.

5. A **multiset** is a set where we allow elements to be picked multiple times. We call a multiset that is a subset of $\mathbb{Z}/(n+1)\mathbb{Z}$ **nullifying** if adding all of its elements together gives us zero (mod $n+1$.) Let C_n denote the total number of n-element nullifying multisets of elements in $\mathbb{Z}/(n+1)\mathbb{Z}$. Here are all of these multisets counted by C_3:

$$\{0,0,0\}, \{0,1,3\}, \{0,2,2\}, \{1,1,2\}, \{2,3,3\}.$$

6. Take all of the sequences of integers $(a_1, a_2, \dots a_n)$ such that

   - $a_i \leq 1$ for every i.
   - Each of the partial sums $\sum_{n=1}^{k} a_i$ is positive, for each $1 \leq k \leq n$.

   Let C_n denote the total number of such sequences of length $n-1$: we give all of the sequences for C_3 here.

$$(0,0), (0,1), (1,-1), (1,0), (1,1).$$

Show that the C_n's satisfy our claimed recurrence.