Math/CS 103	Professor: Padraic Bartlett

Minilecture 5: Affine Planes

Week 4 *UCSB 2014*

1 Affine Planes

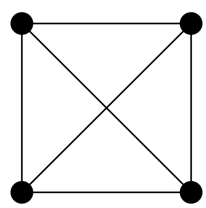
Definition. An **affine plane** is a collection of points and lines in space that follow the following fairly sensical rules:

(A1): Given any two points, there is a unique line joining any two points.

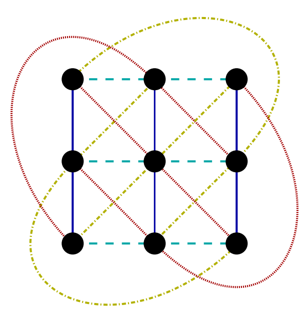
(A2): Given a point P and a line L not containing P, there is a unique line that contains P and does not intersect L.

(A3): There are four points, no three of which are collinear. (This rule is just to eliminate the silly case where all of your points are on the same line.)

$\mathbb{R}^2$ satisfies these properties, and as such is an affine plane. In this class, we're going to be interested in studying **finite** affine planes: i.e. affine planes with finitely many points. For example, the following set of four points and six lines defines an affine plane:



The following set of nine points and twelve lines defines another affine plane:



(All lines in the picture above contain three points. There are three teal dashed lines going left-to-right, three solid blue lines going straight up and down, three gold dash/dot lines on the bottom-left to top-right diagonals, and three maroon small-dash lines on the top-left to bottom-right diagonal.)

Finite affine planes satisfy a number of interesting properties. To get an idea of how these properties work, we study one example here:

Proposition. In any affine plane, there is an integer n such that every line in our plane contains exactly n points, and every point lies on precisely $n+1$ lines. We call this value the **order** of our plane.

Proof. There are two possible cases to consider here:

1. Suppose that for any two lines L_1, L_2 in our plane, we can always find a third point P such that P does not lie on either of these lines. Then, given any point Q on the line L_1, we can find a line M through Q and P using property $A1$ of our affine plane. This line cannot intersect any other elements on L_1, because otherwise (if it did, at some point R) we would not have a unique line defined by the points Q and R (as both L_1 and M would contain both of them, while being distinct lines because M contains P while L_1 does not.)

   So, every point in L_1 is contained within exactly one line through P. Furthermore, there is exactly one other line that goes through P that intersects no point of L_1, by property $A2$. So, if $|L_1|$ denotes the total number of points contained in the line L_1, we have that $|L_1|+1$ many lines go through P.

   Similarly, every point in L_2 is contained within exactly one line through P, and there is precisely one other line through P that does not intersect L_2. Therefore, if $|L_2|$ denotes the total number of points contained in the line L_2, we also have that $|L_2|+1$ many lines go through P.

   But these two things are counting the same object: the number of lines through P. Therefore, these two quantities are equal: i.e. $|L_1| = |L_2|$. Therefore, all lines contain the same number (call it n) of points, and any point is contained by $n+1$ many lines.

2. If we are not in the first case, then there are two lines L_1, L_2 such that every point of P is contained within our two lines. We claim that our plane is in fact the affine plane with four elements that we gave an example of earlier.

   To see why: first, notice that by our property $A3$, we must have two of these points on L_1 and not on L_2, and the other two on L_2 and not on L_1. Call the L_1 points P_1, P_2 and the L_2 points Q_1, Q_2. Suppose for contradiction that we had a third point running around. By assumption, it has to lie on either line L_1 or L_2: without loss of generality, assume it lies on line L_1, and call it P_3.

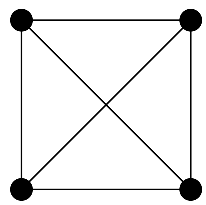
   Examine the line M_1 formed by the points P_1, Q_1, M_2 formed by the points P_2, Q_2 and M_3 formed by the points P_3, Q_2. Note that neither M_1 nor M_2 can be L_1 or L_2, by the argument we just made above.

   We know that at most one of the line M_2, M_3 can be parallel to M_1; therefore, at least one of them must intersect M_1.

   Suppose that M_1 and M_2 intersect at some point. If it is a point P_i on L_1, then the pair of points P_1, P_i defines both of the distinct lines M_1 and L_1, which contradicts our property $A1$. Instead, if it's a point Q_j on L_2, then the pair of points Q_1, Q_j also defines both of the distinct lines M_1 and L_2, which contradicts $A1$ again.

So we cannot have M_1 and M_2 intersecting; therefore, we must have M_1 and M_3 intersect. But this creates the same set of problems — no matter how they intersect, we'll get a pair of points that define two distinct lines!

Therefore, we must have that L_1 and L_2 must contain exactly two points, as must all other lines; consequently, we have that there are four points in total in our plane. Because any two of them uniquely define a line, we have $\binom{4}{2} = 6$ many edges in total, $\binom{3}{1} = 3$ of which pass through any point. This is in particular the affine plane we drew earlier with four points and six edges, which we call the affine plane of order 2.



$\square$

Using this, on the HW you're asked to prove the following:

Proposition. Any finite affine plane of order n contains n^2 many points.

For our third property, the definition of a **parallel class** will be useful:

Definition. A **parallel class** in an affine plane is a collection of lines that are all parallel: i.e. such that no two of them intersect.

Proposition. Take any finite affine plane of order n. Then there are exactly $n^2 + n$ lines in this plane, which can be partitioned into $n + 1$ distinct parallel classes, each of which contains n lines.

This proposition is also on the HW!