Math/CS 103	Professor: Padraic Bartlett

Minilecture 2: Graph Theory

Week 1 | *UCSB 2014*

Graphs are incredibly useful and beautiful objects, used by mathematicians to model groups, electrical networks, airplane routes, and the internet itself! We define them here.

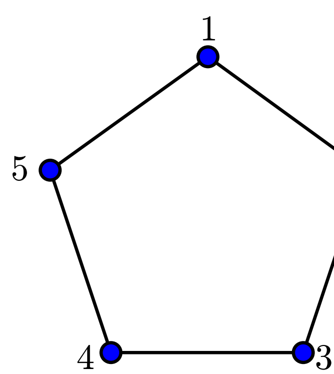
Definition. A graph G with n vertices and m edges consists of the following two objects:

1. a set $V = \{v_1, \ldots v_n\}$, the members of which we call G's **vertices**, and
2. a set $E = \{e_1, \ldots e_m\}$, the members of which we call G's **edges**, where each edge e_i is an unordered pair of distinct elements in V, and no unordered pair is repeated. For a given edge $e = \{v, w\}$, we will often refer to the two vertices v, w contained by e as its endpoints.

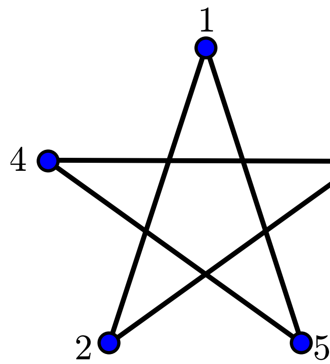
Example. The following pair (V, E) defines a simple graph G on five vertices and five edges:

- $V = \{1, 2, 3, 4, 5\}$,
- $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$.

Something mathematicians like to do to quickly represent graphs is **draw** them, which we can do by taking each vertex and assigining it a point in the plane, and taking each edge and drawing a curve between the two vertices represented by that edge. For example, one way to draw our graph G is the following:

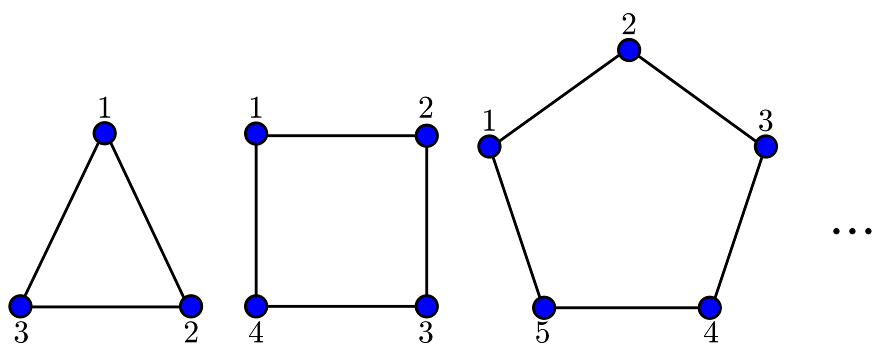


However, this is not the only way to draw our graph! Another equally valid drawing is presented here:

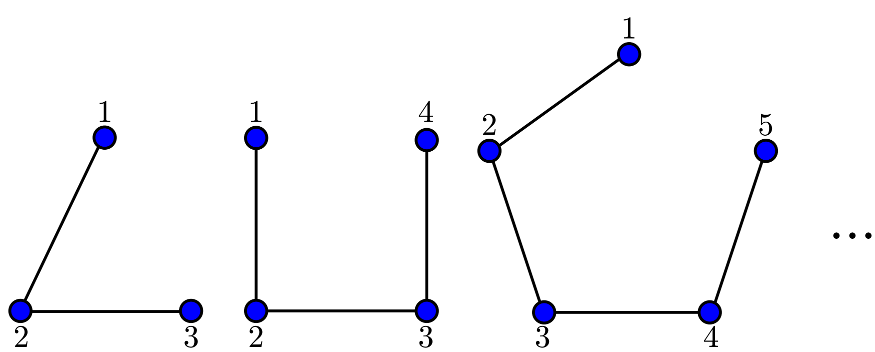


In the field of graph theory, there are several graphs which come up enough to where we've given them names. We list them here, along with some of their properties:

- **The cycle graph C_n.** The cycle graph on n vertices, C_n, is the simple graph on the vertex set $\{v_1, v_2, \dots v_n\}$ with edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. The cycle graphs C_n can be drawn as n-gons, as depicted below:

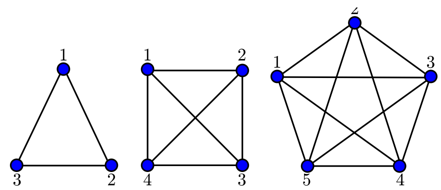


- **The path graph P_n.** The path graph on n vertices, P_n, is the simple graph on the vertex set $\{v_1, v_2, \dots v_n\}$ with edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots \{v_{n-1}, v_n\}\}$. The path graphs P_n can be drawn as paths of length n, as depicted below:



Every vertex in a P_n has degree 2, except for the two endpoints v_1, v_n, which have degree 1. P_n contains $n-1$ edges.

- **The complete graph K_n.** The complete graph on n vertices, K_n, is the simple graph on the vertex set $\{v_1, v_2, \ldots v_n\}$ that has every possible edge: in other words, $E(K_n) = \{\{v_i, v_j\} : i \neq j\}$. We draw several of these graphs below:



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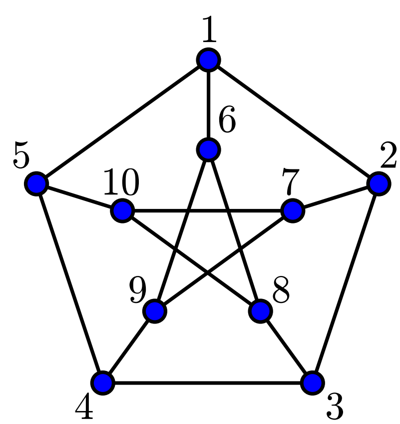
Every vertex in a K_n has degree $n-1$, as it has an edge connecting it to each of the other $n-1$ vertices; as well, a K_n has $n(n-1)/2$ edges in total in it, by the degree-sum formula. (Explicitly: every vertex has degree $n-1$ and there are n vertices, therefore the sum of the degrees of K_n's vertices is $n(n-1)$. We've shown that this quantity is twice the number of edges in the graph; dividing by 2 then tells us that the number of edges in K_n is $n(n-1)/2$, as claimed.)

- **The complete bipartite graph $K_{n,m}$.** The complete bipartite graph on $n+m$ vertices with part sizes n and m, $K_{n,m}$, is the following graph:

  - $V(K_{n,m}) = \{v_1, v_2, \ldots v_n, w_1, w_2, \ldots w_m\}$.
  - $E(K_{n,m})$ consists of all of the edges between the n-part and the m-part; in other words, $E(K_{n,m}) = \{(v_i, w_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$.

The vertices v_i all have degree m, as they have precisely m edges leaving them (one to every vertex w_j); similarly, the vertices w_j all have degree n. By either the degree-sum formula or just counting, we can see that there are nm edges in $K_{n,m}$.

- **The Petersen graph P** The Petersen graph P is a graph on ten vertices, drawn below:

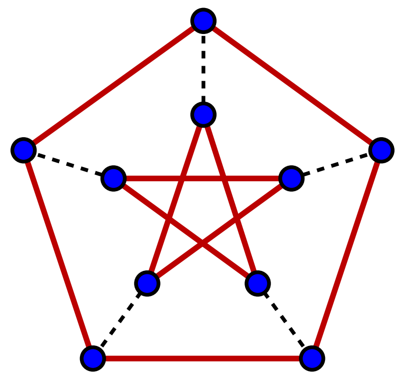


The vertices in P all have degree three; by counting or the degree-sum formula, P has 15 edges.

One useful notion, when talking about graphs, is the idea of a subgraph:

Definition. Given a graph G and another graph H, we say that H is a **subgraph** of G if and only if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

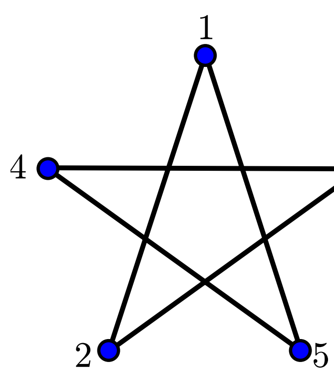
Example. The Petersen graph has the disjoint union of two pentagons $C_5 \sqcup C_5$ as a subgraph, which we shade in red below:



Graphs in general – as you may have noticed – can be remarkably hairy and complicated things. As such, we might wonder whether we can say anything about a general graph at all; in other words, if G is a graph, can we say anything that restricts G?

As it turns out, we can! Consider the following definition:

Definition. In a simple graph G, we say that a vertex $v \in V(G)$ has **degree** k iff there are exactly k edges in $E(G)$ that involve v. For example, in the graph drawn below, every vertex has degree 2:

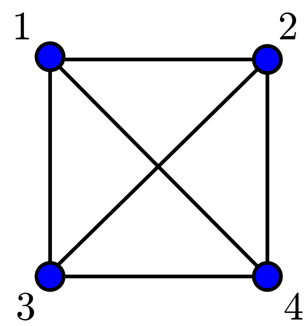


One quick question we could ask: for what values of n can we make a graph with the degrees of all vertices $= 3$?

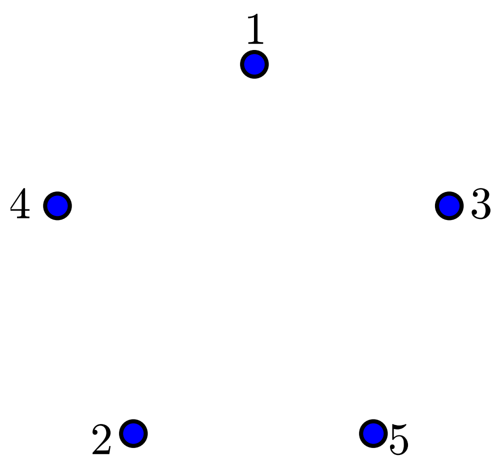
Well: in order to have each vertex have three neighbors[1], we need to have at least four vertices (as each vertex needs three other vertices to send edges to.)

In the case where $n = 4$, we can make every vertex have degree 3, via the graph below:

[1]We say that the vertices v and w are neighbors iff there is an edge connecting v and w. We denote the set of all neighbors of v as $N(v)$.



How about for $k = 5$? Try it for yourself, on the vertices below:



As you may have noticed, no matter what you try, you can't get all of the degrees to be three! As it turns out, this is completely impossible – there is no graph on 5 vertices for which all of the vertices have degree 3!

We explore this question, and others, in today's handout.