Math/CS 103	Professor: Padraic Bartlett

Minilecture 1: Latin Squares

Week 1 *UCSB 2014*

This is a brief introduction to an object that I spent the bulk of my Ph.D studying: **Latin squares**!

Definition. A **latin square** of order n is a $n \times n$ array filled with n distinct symbols (by convention $\{1, \ldots n\}$), such that no symbol is repeated twice in any row or column.

Example. Here are all of the latin squares of order 2:

1	2
2	1

2	1
1	2

.

A quick observation we should make is the following:

Proposition. Latin squares exist for all n.

Proof. Behold!

1	2	$\ldots$	$n-1$	n
2	3	$\ldots$	n	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
n	1	$\ldots$	$n-2$	$n-1$

$\square$

Given this observation, a natural question to ask might be "How many Latin squares exist of a given order n?" And indeed, this is an excellent question! So excellent, in fact, that it turns out that we have no idea what the answer to it is; indeed, we only know the true number of Latin squares of any given order up to 11.

n	reduced Latin squares of size n[1]	all Latin squares of size n
1	1	1
2	1	2
3	1	12
4	4	576
5	56	161280
6	9408	812851200
7	16942080	61479419904000
8	535281401856	108776032459082956800
9	377597570964258816	5524751496156892842531225600
10	7580721483160132811489280	9982437658213039871725064756920320000
11	5363937773277371298119673540771840	776966836171770144107444346734230682311065600000
12	?	?

Asymptotically, the best we know (and you could show, given a lot of linear algebra tools) that

$$L(n) \sim \left(\frac{n}{e^2}\right)^{n^2}.$$

Instead of this question, we're going to spend this class studying the concept of **partial Latin squares**, which we define below:

1 Partial Latin Squares

Definition. A **partial latin square** of order n is a $n \times n$ array where each cell is filled with either blanks or symbols $\{1, \ldots n\}$, such that no symbol is repeated twice in any row or column.

Example. Here are a pair of partial 4×4 latin squares:

			4
2			
3	4		
4	1	2	

1			
	1		
		1	
			2

The most obvious question we can ask about partial latin squares is the following: when can we complete them into filled-in latin squares? There are clearly cases where this is possible: the first array above, for example, can be completed as illustrated below.

			4
2			
3	4		
4	1	2	

$\mapsto$

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

However, there are also clearly partial Latin squares that cannot be completed. For example, if we look at the second array

1			
	1		
		1	
			2

,

we can pretty quickly see that there is no way to complete this array to a Latin square: any 4×4 Latin square will have to have a 1 in its last column somewhere, yet it cannot be in any of the three available slots in that last column, because there's already a 1 in those three rows.

[1] A **reduced** Latin square of size n is a Latin square where the first column and row are both $(1, 2, 3 \ldots n)$. The idea here is that by permuting the rows and columns of any Latin square, you can make it have this "reduced" property. Therefore, in a sense, the only interesting things to count are the number of different reduced squares; this is because from there you can generate any other Latin square by permuting its rows and columns.