Math/CS 103	Professor: Padraic Bartlett

Handout 3: Graphs

Week 1	*UCSB 2014*

In this handout, we are studying a number of questions about **graphs**! These are to be turned in Wednesday, January 22nd.

1. A graph G is said to be a **tree** if it satisfies the following two properties:

   - G is **connected**: i.e. given any two vertices in G, there is a path that is a subgraph of G connecting those two vertices.
   - G does not contain any cycles C_k as subgraphs, for any $k \geq 3$.

   Suppose that G is a connected graph on n vertices with $n-1$ edges. Prove that G is a tree.

2. A graph G is called **bipartite** if it satisfies the following property:

   - The vertices of G can be split into two sets, V_1 and V_2, such that there are no edges in G connecting two vertices in V_1, or connecting two vertices in V_2 (i.e. all edges involve exactly one vertex in V_1 and one in V_2.)

   Prove that G is bipartite if and only if it does not contain any odd cycles as subgraphs.

3. Suppose that a graph G has m edges. A m**-vertex-labeling** of this graph is a way to assign the numbers $\{0, \ldots m\}$ to the vertices of G. Given any such vertex labeling, we get an **induced** m-labeling of the edges of this graph as follows: if an edge $e = \{x, y\}$ has x labeled l_x, y labeled l_y, we can label the edge $e = |l_x - l_y|$.

   We call a m-vertex-labeling **graceful** if

   - No two vertices have the same label.
   - No two edges have the same label in the induced labeling described above.

   Prove that all paths P_n have graceful labelings.

4. (SUPER BONUS QUESTION): Prove that all trees have graceful labelings.